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PARTITION PROPERTY OF DOMAIN DECOMPOSITION WITHOUT ELLIPTICITY*[1)]

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Abstract

Partition property plays a central role in domain decomposition methods. Existing theory essentially assumes certain ellipticity. We prove the partition property for problems without ellipticity which are of practical importance. Example applications include implicit schemes applied to degenerate parabolic partial differential equations arising from superconductors, superfluids and liquid crystals. With this partition property, Schwarz algorithms can be applied to general non-elliptic problems with an h-independent optimal convergence rate. Application to the time-dependent Ginzburg-Landau model of superconductivity is illustrated and numerical results are presented.

Key words: Partition property, Domain decomposition, Non-ellipticity, Degenerate parabolic problems, Time-dependent Ginzburg-Landau model, Superconductivity, Preconditioning, Schwarz algorithms.

1. Introduction

Domain decomposition methods have undergone great development in the past decade and there has been rich mathematical theory for model problems. It is worthy to examine how these methods can be effectively applied to practical problems where the model problem analysis does not trivially apply. It is far from claiming that there remains not much new for domain decomposition theoretically and practically. In terms of domain partition, domain decomposition methods are classified as two types: overlapping and non-overlapping. Due to P. L. Lions [7], a general framework based on projection operators has been established for the convergence analysis of overlapping methods, where a so-called *partition property* play a central role. Following this approach, various overlapping methods such as the multiplicative and additive Schwarz algorithms have been studied for the model elliptic and parabolic problems as well as their extension to nonlinear, non-selfadjoint, and indefinite problems. For each case, the essential task is to verify the validity of the partition property associated with the particular problem. Although mathematically the fundamental analysis framework remains more or less unchanged, these works are of practical importance because they make the application of domain decomposition broader and broader, and more and more practical. There is vast literature available

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in this field, and a complete survey and references are recently found in [5], from which we cite only a few when necessary.

In this paper, we are interested in problems without ellipticity, such as degenerate parabolic PDEs which are not toy problems, nor trivial extension of the parabolic model problem for which the overlapping methods have already been well understood. The study is motivated by superconductor simulation using the three-dimensional time-dependent Ginzburg-Landau model (TDGL), which involves solution to a degenerate parabolic system of coupled nonlinear PDEs [8] [10]. An implicit scheme is used due to the stability consideration. To simulate the superconducting vortex dynamics over a long time period and the vortex structures at the equilibrium state, a fairly large step size Δt has to be used. On the other hand, a small spacing h is required for adequate resolution. This implies that the condition number of the discrete system cannot be improved by decreasing Δt. To simulate a larger sample with more vortices, one must use many more spatial grid points, which in turn implies larger condition number and much more intensive computation which takes considerable time even on IBM SP2 and Intel PARAGON parallel supercomputers. Apparently, preconditioning is necessary and crucial for reducing the computational time of iterative solvers in order to perform practical simulations. In fact, this type of degenerate parabolic problems also occur in other important applications such as superfluids and liquid crystals. We apply overlapping methods to speed up the iteration and prove the optimal convergence rate for these methods. As for other problems mentioned earlier, the key task is to verify that the partition property also holds for this type of problems under the weak assumption. We show that these methods are indeed numerically effective.

The remainder of the paper is organized as follows. We outline P. L. Lions' general framework and describe the partition property in Section 2. In Section 3, the partition property for problems without ellipticity is proved. The convergence theory developed in Section 3 is applied to the Ginzburg-Landau model in Section 4 and numerical results are presented.

2. Partition Property

Starting with a non-overlapping quasi-uniform partition

$$\overline{\Omega} = \bigcup_{i=1}^{I} \overline{\Omega}_i^0 \tag{2.1}$$

of coarse mesh size H, we define an overlapping domain decomposition by extending each subdomain Ω_i^0 to Ω_i such that

$$dist\left(\partial\Omega_i \bigcap \Omega, \partial\Omega_i^0 \bigcap \Omega\right) \geq \beta H, \quad i = 1, \cdots, I, \tag{2.2}$$

where β measures the overlapping width and that for any point $x \in \Omega$, there exist at most N_c subdomains Ω_i containing x. Let V be a finite element space corresponding to a quasi-uniform fine mesh of size h, and assume that the boundaries of the overlapping subdomains align with the fine mesh lines. Associated with the domain decomposition, we define subspaces $V_i \subset V (1 \leq i \leq I)$ as

$$V_i = \{v \in V;\ v = 0 \text{ in } \Omega/\Omega_i\}.$$

One can verify that $V = \sum_{i=1}^{I} V_i$. In addition, one often needs to use V_0 which denotes the finite element space corresponding to the coarse mesh (2.1).