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IMPROVED ERROR ESTIMATES FOR MIXED FINITE ELEMENT FOR NONLINEAR HYPERBOLIC EQUATIONS: THE CONTINUOUS-TIME CASE*1)

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Abstract

Improved L^2-error estimates are computed for mixed finite element methods for second order nonlinear hyperbolic equations. Results are given for the continuous-time case. The convergence of the values for both the scalar function and the flux is demonstrated. The technique used here covers the lowest-order Raviart-Thomas spaces, as well as the higher-order spaces. A second paper will present the analysis of a fully discrete scheme (Numer. Math. J. Chinese Univ. vol.9, no.2, 2000, 181-192).

Key words: Nonlinear hyperbolic equations, Mixed finite element methods, Error estimates, Superconvergence.

1. Introduction

Let Ω be a bounded domain in $\mathbf{R}^2$ with Lipschitz boundary $\partial\Omega$, and unit outward normal ν. For fixed $0 < T < \infty$, $J = (0, T]$, we discuss mixed finite element approximations of second order nonlinear hyperbolic equation

$$c(x,u)u_{tt} - \nabla\cdot(a(x,u)\nabla u) = f(x,u,t), \qquad x\in\Omega, \quad t\in J, \tag{1.1}$$

with initial conditions

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x\in\Omega, \tag{1.2}$$

and Dirichlet boundary condition

$$u(x,t) = -g(x,t), \qquad (x,t)\in\partial\Omega\times J. \tag{1.3}$$

We shall assume that the functions $c(x,u)$, $a(x,u)$, $f(x,u,t)$, $g(x,t)$ and solution $u(x,t)$ have sufficient regularity. Additionally, we assume that there exist constants c_*, c^*, a_*, and a^* such that

$$0 < c_* \le c(x,u) \le c^*, \qquad 0 < a_* \le a(x,u) \le a^*, \tag{1.4}$$

Optimal rates of convergence for Galerkin approximations to a class of second order nonlinear hyperbolic equations have been previously derived by Yuan yi-rang and Wang hong [9, 12-13]. The study of superconvergence for the gradient of the solution of second order hyperbolic equation was provided in [1, 6, 10-11]. Recently, several works have been devoted to the analysis of the mixed finite element methods (see [2-5, 8]). Cowsar, Dupont, Wheeler [2] have considered the convergence of the mixed finite element methods for second order linear hyperbolic equation.

In this paper, we formulate a mixed finite elment scheme for the approximation of (1.1)-(1.3) and establish the superconvergence L^2-estimate between the finite element solution and its elliptic projection. The method here gives a direct approximation of the flux, rather than

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one that requires differentiation and multiplication by a possibly rapidly varying coefficient; so, the direct evaluation of the flux can be expected to give improved accuracy for the same computational effort.

2. Mixed Finite Element Formulation for Nonlinear Hyperbolic Problem

Let $V = H(\text{div};\Omega)$, $W = L^2(\Omega)$. Introduce the flux variable z:

$$z = -a(x,u)\nabla u, \tag{2.1}$$

and let $\alpha(u) = \alpha(x,u) = 1/a(x,u)$, $c(u) = c(x,u)$, $f(u) = f(x,u,t)$. Before defining a mixed finite element procedure we rewrite (1.1)-(1.3) in the following weak formulation

$$(u(0), w) = (u_0, w), \qquad w \in W, \tag{2.2}$$

$$(u_t(0), w) = (u_1, w), \qquad w \in W, \tag{2.3}$$

$$(c(u)u_{tt}, w) + (\nabla\cdot z, w) = (f(u), w), \qquad w \in W, \tag{2.4}$$

$$(\alpha(u)z, v) - (\nabla\cdot v, u) =< g, v\cdot\nu >, \qquad v \in V, \tag{2.5}$$

where $(\cdot,\cdot)$ is $L^2(\Omega)$ inner product, $<\cdot,\cdot>$ is the $L^2(\partial\Omega)$ inner product.

For h a small positive parameter we take $W_h \times V_h \subset W \times V$ to be the Raviart-Thomas space [8] of index k, where k is fixed nonnegative integer, associated with $\mathcal{T}_h$.

The continuous-time mixed finite element approximation to (2.2)-(2.5) is defined as a map from $[0,T]$ into $W_h \times V_h$ given by the pair $(U(\cdot,t), Z(\cdot,t))$ satisfying

$$(c(U)U_{tt}, w) + (\nabla\cdot Z, w) = (f(U), w), \qquad w \in W_h, \tag{2.6}$$

$$(\alpha(U)Z, v) - (\nabla\cdot v, U) =< g, v\cdot\nu >, \qquad v \in V_h, \tag{2.7}$$

with initial conditions

$$U(0) = \widetilde{U}(0), \qquad U_t(0) = \widetilde{U}_t(0), \qquad Z(0) = \widetilde{Z}(0), \tag{2.8}$$

where $(\widetilde{U}(\cdot,t), \widetilde{Z}(\cdot,t))$ is the elliptic mixed method projection to be defined later.

3. Mixed Method Projection

For the solution $u(x,t)$, let $\alpha_1 = \alpha(u)$, $\gamma(u) = \alpha_u(u)z$, $\alpha_u(u) = \frac{\partial\alpha(u)}{\partial u}$, and $\gamma_1 = \gamma(u)$. Define a linear mixed elliptic projection of $W \times V$ onto $W_h \times V_h$ by the $(u,z) \to (\widetilde{U}, \widetilde{Z})$ determined by the relations

$$\begin{aligned} &(\nabla\cdot(z-\widetilde{Z}), w) + \lambda(u-\widetilde{U}, w) = 0, && w \in W_h,\\ &(\alpha_1(z-\widetilde{Z}), v) - (\nabla\cdot v, u-\widetilde{U}) + (\gamma_1(u-\widetilde{U}), v) = 0, && v \in V_h, \end{aligned} \tag{3.1}$$

for each $t \in J$. The positive constant λ will be assumed to be a sufficiently large constant such that

$$(\alpha_1\zeta,\zeta) + \lambda(\xi,\xi) + (\gamma_1\xi,\zeta) \geq \lambda_0(||\zeta||_0^2 + ||\xi||_0^2), \quad \text{for } \zeta \in V \text{ and } \xi \in W, \tag{3.2}$$

where $\lambda_0 > 0$ is independent of $t \in J$. Let

$$\begin{aligned} \eta &= u - \widetilde{U}, & \rho &= z - \widetilde{Z},\\ \xi &= \widetilde{U} - U, & \zeta &= \widetilde{Z} - Z. \end{aligned} \tag{3.3}$$

As shown in [3, 4, 8], there exist the Raviart-Thomas projection $\Pi_h: \; V \to V_h$ and L^2-projection $P_h: \; W \to W_h$ such that for $0 < q \leq \infty$,

$$\text{div}\circ\Pi_h = P_h\circ\text{div}, \tag{3.4}$$