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CONSTRUCTION OF A PRECONDITIONER FOR DOMAIN DECOMPOSITION METHODS WITH POLYNOMIAL LAGRANGIAN MULTIPLIERS*1)

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Abstract

In this paper we consider domain decomposition methods with polynomial Lagrangian multipliers to two-dimentional elliptic problems, and construct a kind of simple preconditioners for the corresponding interface equation. It will be shown that condition number of the resulting preconditioned interface matrix is almost optimal (namely, it has only logarithmic growth with dimension of the local interface space).

Key words: Domain Decomposition, Non-matching grids, Lagrangian multipliers, Preconditioner, Condition number.

1. Introduction

In recent years the non-overlapping domain decomposition methods (DDMs) with non-matching grids have attracted particular attention of computional experts and engineers (see [1]-[9]). This kind of DDM allows non-coincidence of nodal points at common edges (or common faces) of two neighbouring subdomains. Thus it can be applied to solving the problems of changing meshes (for example, the multi-body contact problems in solid mechanics and the relative motion problems in oil exploration), and can be applied to designing the optimal meshes, namely, one can choose different mesh-sizes and different orders of approximate polynomials in different subdomains according to different properties of solutions and different requirements of practical problems.

There are three kinds of important algorithms to deal with the interface non-conformity generated by the non-matching grids, namely, the mortar element method (see [1]-[3], [8]-[10]), the Lagrangian multipliers method (see [4], [5], [10]-[14]) and the augmented Lagrangian method (see [9]). For the mortar element method, the interface variable is chosen as a proper approximation of the trace of numerical solution on the interface, thus it is a direct extention of the usual non-overlapping DDM. For the Lagrangian multipliers method, the interface variable (namely, the Lagrangian multiplier) is chosen as a proper approximation of the normal derivative on the interface, which transforms the minimization problem with restriction (namely, weak continuity of the trace on the interface) into the corresponding saddle-point problem without restriction, thus it is the dual algorithm of the mortar element method. The augmented Lagrangian method can be understood as a mixed algorithm generated by combining the mortar element method with the Lagrangian multiplier method.

The DDM with Lagrangian multiplier (DDMLM) has obvious advantages over the mortar element DDM: (a) the interface variable associated with DDMLM need not be continuous at the cross-points (for the case of two-dimension) or on the cross-edges (for the case of three-dimension), so the corresponding interface equation can be formed easily; (b) the construction of the interface subspace associated with the DDMLM is flexible, thus the non-matching grids do not bring about any difficulty; (c) the DDMLM may reduce the size of the interface problem.

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In [5], we introduced and analysed a kind of DDMLM in which the space of the Lagrangian multipliers consists of polynomials of the certain degree n (in [11] and [12], this method was mentioned too). This method has advantages in comparison with another kind of DDMLM in which special partitioning of the interface is introduced and the space of the Lagrangian multiplier is chosen as the corresponding finite element space (refer to [1], [9], [10] and [12]-[14]): (i) the numerical integrations defined on the interface can be calculated conveniently; (ii) the size of the interface problem can be reduced greatly when the exact solution has good smoothness on the interface. However, for this kind of DDMLM condition number of the interface matrices is highly sensitive to the number n.

It is well known that, for non-overlapping DDMs, construction of interface preconditioners is a core problem. From the advantage (a) mentioned above, we know that construction of interface preconditioner associated with DDMLM is essentially different from the case of the usual non-overlapping DD method (how to construct coarse subspace ?).

In this paper we advanced a new idea (refer to [19] and [20]) in which the coarse subspace consists of piecewise contants. Based on this, we construct a kind of preconditioner for the DDMLM to two-dimensional elliptic problems, and show that condition number of the correspondding preconditioned interface matrix is almost optimal. The preconditioner proposed in here has obvious advantages: (i) it is independent of the cross-points, thus the computational procedure is easy to design (in comparison with the preconditioners constructed in [15] and [16]); (ii) it is independent of the trace space, thus the problems of changing meshes do not bring about any difficulty (note that all the preconditioners introduced in [4], [13] and [14] depend on the usual Scur complement); (iii) the local solvers are defined on the common edges of two neighbouring subdomains, thus it results in cheap calculation (in comparison with the preconditioners discussed in [8], [13], [14], [17] and [18]);

The idea advanced in this paper is also fit for three-dimensional elliptic problems (see [19]).

2. The DDMLM

For ease of notation, we consider the following model problems:

$$\begin{cases} -\Delta u + \eta u = f, & in\ \Omega, \\ u = 0, & on\ \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset R^2$ is a polygonal domain, and η is a positive number which is bounded above.

The domain Ω is decomposed into N polygons Ω_i. We first make the following assumptions:
H_1: all subdomain Ω_i are of size d in the sense that there exists constants c_0 and c_1 independent of d such that each Ω_i contains (resp.is contained in) a circle of radius $c_0 d$ (resp.$c_1 d$);
H_2 :For $i \neq j$, we require that if two (open) edges $F' \subset \partial\Omega_i$ and $F'' \subset \partial\Omega_j$ share a common point, then $\overline{F}' = \overline{F}'' = \overline{\Omega}_i \bigcap \overline{\Omega}_j = F_{ij}$. Let F=$\bigcup F_{ij}$ denotes the entire interface;
H_3: each subdomain Ω_i consists of quasi-uniform triangular or quadrilateral elements with diamenter h_i.

For a natural number n, and $h = \max_{1\leq i\leq N} h_i$, we assume that
$H_4 : \lim_{h\to 0}(n^2 h/d) = 0$.

Since N is in general large, namely, d is small, thus we can assume that
$H_5 : \eta \leq C d^{-2}$.

Now, we define the approximation spaces as follows:

Let $S_{h_i}(\Omega_i)$ be the space of continuous piecewise m_i degree polynomials defined on Ω_i. Set

$$S_h(\Omega) = \{\varphi : \varphi\mid_{\partial\Omega} = 0, \varphi\mid_{\Omega_i} \in S_h(\Omega_i), i = 1, \cdots, N\}$$

$$S_n(F_{ij}) = \{\lambda_{ij} : \lambda_{ij}\ is\ a\ polynomial\ of\ degree \leq n\ on\ F_{ij}\}$$

$$S_n(\partial\Omega_i) = \{\lambda_i : \lambda_i\mid_{F_{ij}} \in S_n(F_{ij}), F_{ij} \subset \partial\Omega_i\}$$

$$S_n(F) = \{\lambda : \lambda\mid_{F_{ij}} \in S_n(F_{ij})\ for\ all\ F_{ij}\}$$

$$S_{h\times n} = S_h(\Omega) \times S_n(F)$$

Remark 1. The boundary nodes of the triangulation of Ω_i and Ω_j need not coincide on the common edges (namely, we have not imposed any matching conditions for the grids at