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CONVERGENCE AND SUPERCONVERGENCE OF HERMITE BICUBIC ELEMENT FOR EIGENVALUE PROBLEM OF THE BIHARMONIC EQUATION*

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Abstract

In this paper, we discuss the convergence and superconvergence for eigenvalue problem of the biharmonic equation by using the Hermite bicubic element. Based on asymptotic error expansions and interpolation postprocessing, we gain the following estimation:

$$0 \leq \overline{\lambda}_h - \lambda \leq C_\epsilon h^{8-\epsilon}$$

where $\epsilon > 0$ is an arbitrary small positive number and $C_\epsilon > 0$ is a constant.

Key words: Hermite bicubic element, Biharmonic equation, Interpolation postprocessing, Eigenvalue problem.

Eigenvalue problem for biharmonic equation is an interesting and important problem, see Ciarlet and Lions[3]. In 1979, Rannacher[8] used the Adini nonconforming finite element to solve this problem and obtained:

$$|\lambda_h - \lambda| \leq Ch^2.$$

Recently, Yang[6] has proved that the order of covergence of λ_h is just 2. The aim of this paper is to improve the order of convergence by using Hermite bicubic element. To our knowledge, there is not any result for approximation to the eigenvalue problem by using this element in literature. This paper fills in this gap. Not only the convergence is discussed:

$$0 \leq \lambda_h - \lambda \leq Ch^4.$$

but also a superconvergence is presented:

$$0 \leq \overline{\lambda}_h - \lambda \leq C_\epsilon h^{8-\epsilon}.$$

where $\epsilon > 0$ is an arbitrary small positive number.

Note that the degree of freedom is 12 for Adini element and 16 for Hermite bicubic element, respectively. We can see that the increase of the degree of freedom for Hermite element is not too much, but the order of convergence and superconvergence by using Hermite element is one times and even two times higher than that of Adini element scheme, respectively. We mention that our techniques here are based on asymptotic error expansions and interpolation postprocessing.[1,2,5,7]

Let Ω be a rectangle domain. We will consider the following eigenvalue problem of biharmonic equation :

$$\begin{cases} \Delta^2 u = \lambda u & \text{in} \quad \Omega \\ u|_{\partial\Omega} = \frac{\partial u}{\partial n}|_{\partial\Omega} = 0 \end{cases}$$

Its weak form reads as:find a pair of $(\lambda, u), \lambda \in R, u \in H_0^2(\Omega), ||u||_0 = 1$ such that

$$a(u,v) = \lambda(u,v), \qquad \forall v \in H_0^2(\Omega), \tag{1}$$

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where

$$a(u,v)=\int_\Omega (u_{xx}v_{xx}+2u_{xy}v_{xy}+u_{yy}v_{yy})dxdy.$$

Let $T_h=\{e\}$ be a uniform rectangular partition of Ω and $V_h\subset H_0^2(\Omega)$ be the Hermite bicubic finite element space associated with T_h. We use $Q_p(e)$ represent the complete polynomial of degree p. $v\in V_h$ if and only if

1) $v|_e\in Q_3(e), \qquad \forall e\in T_h$;

2) v,v_x,v_y,v_{xy} are continuous at the vertices and vanish at the vertices along $\partial\Omega$.

The finite element scheme is: find a pair of $(\lambda_h,u_h),\lambda_h\in R,u_h\in V_h,||u_h||_0=1$, such that

$$a(u_h,v)=\lambda_h(u_h,v), \qquad \forall v\in V_h. \tag{2}$$

Let (λ,u) and (λ_h,u_h) denote the exact eigenpair of biharmonic operator and the associated approximate eigenpair in the finite element space V_h,respectively. $R_hu\in V_h$ represents the Ritz-Galerkin projection of eigenfunction u. $C>0$ acts as a positive constant but it can take different value in this paper. $C_\epsilon>0$ is a positive number which depends on ϵ. It is well known that[1,3,5]:

$$\lambda=\inf_{\phi\neq\theta,\phi\in H_0^2}\frac{||\phi||_a^2}{(\phi,\phi)}, \tag{3}$$

$$\lambda_h=\inf_{v\neq\theta,v\in V_h}\frac{||v||_a^2}{(v,v)}, \tag{4}$$

$$||R_hu-u||_2\le Ch^2||u||_4, \tag{5}$$

where

$$||w||_a^2=a(w,w), \qquad \forall w\in H_0^2(\Omega).$$

Proposition 1.

$$0\le\lambda_h-\lambda\le Ch^4,$$
$$||u_h-u||_0\le C_\epsilon h^{4-\epsilon}.$$

where $\epsilon>0$ is an arbitrary small positive number.

Proof. First of all, we prove the first conclusion. Suppose $(\lambda,u),(||u||_0=1)$ be an eigenpair of operator Δ^2. Then we have

$$a(u,v)=\lambda(u,v), \qquad \forall v\in H_0^2(\Omega). \tag{6}$$

obviously

$$(u,v)=0\Longleftrightarrow a(u,v)=0.$$

$\forall w\in H_0^2(\Omega)$, let $v=w-(w,u)u,u_1=(w,u)u$ then $(v,u_1)=0$ and $a(v,u_1)=0$, namely

$$||w||_a^2=||u_1||_a^2+||v||_a^2.$$

Because $||u_1||_a^2=(w,u)^2a(u,u)=\lambda(w,u)^2$, we have the identity as below

$$||w||_a^2=\lambda(u,w)^2+||w-(w,u)u||_a^2, \tag{7}$$

therefore, $\forall w\in H_0^2(\Omega),w\neq\theta$

$$\lambda\le\frac{||w||_a^2}{||w||_0^2}\le\lambda+\frac{||w-(w,u)u||_a^2}{||w||_0^2}. \tag{8}$$

Due to $w-(w,u)u$ and space $H_\lambda\equiv\{\alpha u:\alpha\in R\}$ are orthogonal for energy inner product $a(\cdot,\cdot)$,so

$$||w-(w,u)u||_a^2\le||w-u||_a^2$$

and

$$0\le\frac{a(w,w)}{(w,w)}-\lambda\le\frac{||w-u||_a^2}{(w,w)}, \qquad \forall w\in H_0^2(\Omega). \tag{9}$$