Journal of Computational Mathematics, Vol.19, No.1, 2001, 9–14.

ANALYSIS OF A MECHANICAL SOLVER FOR LINEAR SYSTEMS OF EQUATIONS*[1)]

Luis Vázquez
(*Dept. Matemática Aplicada, Facultad de Informática, Universidad Complutense, 28040-Madrid, Spain*)

Salvador Jiménez
(*Dept. Matemática y Física Aplicadas, Universidad Alfonso X El Sabio, Avda. Universidad 1, 28691-Villanueva de la Cañada, Madrid, Spain*)

Dedicated to the 80th birthday of Professor Feng Kang

Abstract

In this contribution we analyse some fundamental features of an iterative method to solve systems of linear equations, following the approach introduced in a previous work[1]. Such questions range from optimal parameters and initial conditions to comparison with other methods. An interesting result is that *a priori* we can give an estimation of the number of iterations to get a given accuracy.

Key words: Iterative method, Linear systems, Classical dynamics.

1. Introduction

A new approach to solve systems of linear equations, equivalent to solve the motion of a damped harmonic oscillator, has been proposed in a previous paper[1]. Due to this parallelism, we call such methods *Mechanical Solvers* for systems of linear equations. The present study is devoted to the analysis of these methods.

Let be the linear system

$$A\vec{x} = \vec{b} \tag{1}$$

where we assume that A is an $m \times m$ nonsingular matrix (i.e. the system has a unique solution). We may associate to it the Newton's equation for a linear dissipative ($\alpha > 0$) mechanical system:

$$\vec{x}_{tt} + \alpha\vec{x}_t + A\vec{x} = \vec{b}. \tag{2}$$

If A has a positive real spectrum, we have

$$\lim_{t\to\infty} \vec{x}(t) = A^{-1}\vec{b} \tag{3}$$

which is the solution of the linear system (1). Different equations of motion can be proposed for the system above, of the form

$$\vec{x}_{tt} + \alpha\vec{x}_t + M\vec{x} = \vec{v} \tag{4}$$

* Received November 8, 2000.

[1)] This work has been partially supported by the Comisión Interministerial de Ciencia y Tecnología of Spain under grant PB98-0850.

such that:

$$M\vec{x} = \vec{v} \iff A\vec{x} = \vec{b} \tag{5}$$

In order to avoid problems with the spectrum of A, we may choose

$$M = A^{\mathrm{T}}A,\ \vec{v} = A^{\mathrm{T}}\vec{b}. \tag{6}$$

Although this may not be a good idea if A is ill conditionned [2], we ensure that M is symmetric and positive definite by construction and thus has a real, positive definite spectrum. This will be used in what follows.

The next step is to solve the differential equation with a simple finite-difference scheme, such as:

$$\frac{\vec{x}_{n+1} - 2\vec{x}_n + \vec{x}_{n-1}}{\tau^2} + \alpha\frac{\vec{x}_{n+1} - \vec{x}_{n-1}}{\tau} + M\vec{x}_n = \vec{v} \tag{7}$$

Every finite-difference method associated to (4) will define an iterative process to solve the system (5).

2. Analysis of the Numerical Scheme

Although a single equation is more accurate, for the sake of the analysis we translate (7) into a system of two equations. Keeping in mind the Mechanical analogy we define:

$$\vec{p}_n = \frac{\vec{x}_{n+1} - \vec{x}_n}{\tau} \tag{8}$$

with this and (7) the scheme becomes

$$\begin{cases} \vec{x}_{n+1} = \vec{x}_n + \tau\vec{p}_n \\[2ex] \left(\frac{\alpha}{2}I + \tau M\right)\vec{x}_{n+1} + \left(1 + \frac{\tau\alpha}{2}\right)\vec{p}_{n+1} = \frac{\alpha}{2}\vec{x}_n + \vec{p}_n + \tau\vec{v} \end{cases} \tag{9}$$

where I is the $m \times m$ identity matrix. Let us write this in block-matrix form as:

$$\underbrace{\left(\begin{array}{c|c} \frac{\alpha}{2}I + \tau M & \left(1 + \frac{\tau\alpha}{2}\right)I \\ \hline I & \mathcal{O} \end{array}\right)}_{N_+} \underbrace{\left(\begin{array}{c} \vec{x}_{n+1} \\ \hline \vec{p}_{n+1} \end{array}\right)}_{\vec{Y}_{n+1}} = \underbrace{\left(\begin{array}{c|c} \frac{\alpha}{2}I & I \\ \hline I & \tau I \end{array}\right)}_{N_-} \underbrace{\left(\begin{array}{c} \vec{x}_n \\ \hline \vec{p}_n \end{array}\right)}_{\vec{Y}_n} + \underbrace{\left(\begin{array}{c} \tau\vec{v} \\ \hline \vec{0} \end{array}\right)}_{\vec{W}} \tag{10}$$

and define N_+, N_-, $\vec{Y}_{n+1}$, $\vec{Y}_n$ and $\vec{W}$ as indicated in the previous formula. We have thus an iterative process that we may write formally as

$$\vec{Y}_{n+1} = (N_+)^{-1}N_-\vec{Y}_n + (N_+)^{-1}\vec{W} \tag{11}$$

A sufficient condition to ensure the convergence of this process for any initial condition is to have all eigenvalues of

$$N \equiv (N_+)^{-1}N_- \tag{12}$$

of modulus strictly less than 1. Let us compute those eigenvalues:

$$\lambda \text{ is eigenvalue of } N \iff \left|\begin{array}{c|c} (1-\lambda)\frac{\alpha}{2}I - \lambda\tau M & \left[1 - \lambda\left(1 + \frac{\tau\alpha}{2}\right)\right]I \\ \hline (1-\lambda)I & \tau I \end{array}\right| = 0 \tag{13}$$