Journal of Computational Mathematics, Vol.19, No.1, 2001, 87–100.

CONVERGENCE OF AN EXPLICIT UPWIND FINITE ELEMENT METHOD TO MULTI-DIMENSIONAL CONSERVATION LAWS*1)

Jin-chao Xu
(*Center for Computational Mathematics and Applications and Department of Mathematics, Pennsylvania State University, U.S.A*)
(*School of Mathematical Sciences, Peking University, Beijing 100871, China*)

Lung-an Ying
(*School of Mathematical Sciences, Peking University, Beijing 100871, China*)

Dedicated to the 80th birthday of Professor Feng Kang

Abstract

An explicit upwind finite element method is given for the numerical computation to multi-dimensional scalar conservation laws. It is proved that this scheme is consistent to the equation and monotone, and the approximate solution satisfies discrete entropy inequality. To guarantee the limit of approximate solutions to be a measure valued solution, we prove an energy estimate. Then the L^p strong convergence of this scheme is proved.

Key words: Conservation law, Finite element method, Convergence.

1. Introduction

The convergence problem for the numerical schemes to one dimensional conservation laws has been extensively studied. By tensor product one dimensional schemes can be applied to multi-dimensional equations. However the convergence of many of those schemes is still unknown even if it is true for one dimensional cases. Besides, for those physical domains with complicated geometry unstructured grids are more flexible. In recent years the convergence problem for unstructured grids has called the attention of some authors. In [2] [3] [10] [11] the convergence of finite volume methods was proved. In [9] [8] [13] [14] the convergence of a streamline diffusion finite element method was proved. In [6] [5] the discontinuous Galerkin method to multi-dimensional conservation laws was studied. We refer the readers to a survey by Shu [12].

In this paper we prove the convergence of an explicit upwind finite element method, the edge-averaged finite element method, given in [15] for convection-diffusion equations, to multi-dimensional scalar conservation laws. Our technique is the new approach using measure valued solutions (see [4] [13]). However since artificial viscosity is employed in the scheme, the estimates are quite different, and we believe that our technique can be applied to more general schemes. Numerical experiments have shown that this scheme gives satisfactory results, which will be reported later on.

We consider the equation

$$\frac{\partial u}{\partial t} + \nabla \cdot f(u) = 0, \tag{1}$$

where $f : \mathsf{R} \to \mathsf{R}^N (N \geq 1), f \in C^1$. Without loss of generality we assume that $f(0) = 0$, otherwise $f(u)$ can be replaced by $f(u) - f(0)$.

* Received December 9, 1998.
1)The second author was supported by the China State Major Key project for Basic Researches and the Science Fund of the Ministry of Education of China.

2. The Finite Element Scheme and Main Convergence Theorem

Let T be a simplicial finite element with linear shape functions interpolated by $N+1$ vertices, $q_1, \cdots, q_{N+1}$. Then it is easy to prove the following lemmas[15].

Lemma 2.1. *If $u, v \in P_1(T)$, then*

$$\int_T \nabla u \cdot \nabla v \, dx = \sum_{i<j} a_{i\,j}^T (u_i - u_j)(v_j - v_i),$$

where u_i, u_j, v_i, v_j are the values of u, v at nodes q_i and q_j,

$$a_{i\,j}^T = \int_T \nabla \chi_i \cdot \nabla \chi_j \, dx,$$

and $\chi_i \in P_1(T), i = 1, \cdots, N+1$, $\chi_i(x_j) = \delta_i^j$.

Lemma 2.2. *If $v \in P_1(T)$ and c is a constant vector, then*

$$\int_T c \cdot \nabla v \, dx = \sum_{i<j} a_{i\,j}^T c \cdot \tau_E h_{ij} (v_i - v_j),$$

where E is the edge connecting q_i, q_j, τ_E is the unit vector pointing from q_i to q_j, and h_{ij} is the length of E.

By adding an artificial viscosity term equation (1) becomes

$$\frac{\partial u}{\partial t} + \nabla \cdot f(u) = \varepsilon \triangle u, \tag{2}$$

where $\varepsilon > 0$.

The weak formulation of the initial value problem of the equation (2) is: given u^0, find $u : [0, T] \to H^1(\mathsf{R}^N)$ such that

$$\frac{d}{dt} \int_{\mathsf{R}^N} uv \, dx - \int_{\mathsf{R}^N} f(u) \cdot \nabla v \, dx + \varepsilon \int_{\mathsf{R}^N} \nabla u \cdot \nabla v \, dx = 0, \qquad \forall v \in H^1(\mathsf{R}^N), \tag{3}$$

$$\int_{\mathsf{R}^N} u|_{t=0} v \, dx = \int_{\mathsf{R}^N} u^0 v \, dx, \qquad \forall v \in H^1(\mathsf{R}^N).$$

Let

$$J = \varepsilon \nabla u - f(u),$$

then the projection of J on E is

$$J \cdot \tau_E = \varepsilon \frac{\partial u}{\partial \tau_E} - f(u) \cdot \tau_E.$$

We define an integral along E with $x \in E$,

$$\psi = -\int_{q_i}^{x} \frac{f(u) \cdot \tau_E}{\varepsilon u} \, ds, \tag{4}$$

then

$$J \cdot \tau_E = \varepsilon e^{-\psi} \frac{\partial}{\partial \tau_E} (e^{\psi} u).$$

Integrating on E one has

$$(e^{\psi} u)_j - (e^{\psi} u)_i = \int_{q_i}^{q_j} \frac{J \cdot \tau_E}{\varepsilon} e^{\psi} \, ds,$$