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ON THE SOLVABILITY OF GENERAL LINEAR METHODS FOR DISSIPATIVE DYNAMICAL SYSTEMS[*1)]

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Abstract

The main purpose of the present paper is to examine the existence and local uniqueness of solutions of the implicit equations arising in the application of a weakly algebraically stable general linear methods to dissipative dynamical systems, and to extend the existing relevant results of Runge-Kutta methods by Humphries and Stuart(1994).

Key words: General linear methods, Dissipative dynamical systems, Weak algebraic stability, Solvability.

1. Introduction

The numerical approximation of dissipative initial value problems on R^m by fixed time-stepping Runge-Kutta methods has been considered by Humphries and Stuart[1], and it was shown that the numerical solution defined by an algebraically stable method has an absorbing set and is hence dissipative for any fixed step-size $h > 0$. In 1996, Xiao[8] extended the corresponding relevant results in [1], and showed that two classes of algebraically stable general linear methods applied to dissipative dynamical systems on R^m are dissipative and possess an absorbing set. But the results in [8] have an implicit assumption that the implicit equations arising in the application of the general linear method to dissipative dynamical systems are soluble.

The main purpose of the present paper is to examine the existence and local uniqueness of solutions of the implicit equations arising in the application of a weakly algebraically stable general linear methods to dissipative dynamical systems, and to extend the existing relevant results of Runge-Kutta methods by Humphries and Stuart[1].

Consider the dissipative initial value problem on R^m(cf.[1])

$$y'(t) = f(y), \quad t\geq 0; \qquad y(0) = y_0 \in R^N, \tag{1.1}$$

where the map $f : R^N \to R^N$ is assumed to be locally Lipschitz and continuous, and satisfies the following condition:

$$< x, f(x) > \leq \alpha - \beta \|x\|^2, \quad \forall x \in R^m, \tag{1.2}$$

where and throughout the following, $\alpha\geq 0$, $\beta > 0$, $< .,. >$ is the standard inner product

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on R^m with the corresponding norm $\|.\|$ denoted by $\|u\|^2 =< u, u >$. By means of the theory of ordinary differential equations, we can know that the problem (1.1) is locally uniquely soluble with the solution $y(t)$.

The problem (1.1)-(1.2) arises in many applications and the class defined by (1.1)-(1.2) contains many-known problems (such as some forms of Cahn-Hilliand equations, the Navier-Stokes equations in two dimensions, the Lorenz equations, etc.). The problem (1.1)-(1.2) defines a dynamical system on R^m and possesses an absorbing set $B = B(0, (\frac{\alpha}{\beta})^{\frac{1}{2}} + \varepsilon)$ (i.e. an open ball with the radius $(\frac{\alpha}{\beta})^{\frac{1}{2}} + \varepsilon$ and the center 0) for any $\varepsilon > 0$ and a global attractor A defined by $A = \omega(B)$, where $\omega(B)$ is the ω-limit set of B (cf.[1]).

Consider the r-value s-stage general linear method(cf.[4,6,7]) applied to (1.1)

$$\begin{cases} Y_i &= h\sum\limits_{j=1}^{s} c_{ij}^{11} f(Y_j) + \sum\limits_{j=1}^{r} c_{ij}^{12} y_j^{(n-1)}, \qquad i = 1, 2, \cdots, s, \\ y_i^{(n)} &= h\sum\limits_{j=1}^{s} c_{ij}^{21} f(Y_j) + \sum\limits_{j=1}^{r} c_{ij}^{22} y_j^{(n-1)}, \qquad i = 1, 2, \cdots, r, \\ y_n &= \sum\limits_{j=1}^{r} \sigma_j y_j^{(n)}, \end{cases} \tag{1.3}$$

where $h > 0$ is the given stepsize, c_{ij}^{IJ} and σ_j are real constants, the vectors Y_i are the internal stages of the current step and are approximations to $y(t_n + \mu_i h)$; the vectors $y_i^{(n)}$ are the external stages which contain all information from the previous step necessary for the computation of the new approximation and are approximations to $H_i(t_n + \nu_i h)$; y_n approximates to $y(t_n + \eta h)$. $t_n = nh$, μ_i, ν_i and η are real constants, each $H_i(t_n+\nu_i h)$ denotes a piece of information about the true solution $y(t)$. Let

$$\begin{cases} y^{(n)} &= (y_1^{(n)^T}, y_2^{(n)^T}, \cdots, y_r^{(n)^T})^T \in R^{Nr}, \\ Y &= (Y_1^T, Y_2^T, \cdots, Y_s^T)^T \in R^{ms}, \\ F(Y) &= (f^T(Y_1), f^T(Y_2), \cdots, f^T(Y_s))^T \in R^{ms}, \\ C_{IJ} &= [c_{ij}^{IJ}], \qquad \widetilde{C}_{IJ} = C_{IJ} \otimes I_m, \qquad I, J = 1, 2, \\ \sigma &= (\sigma_1, \sigma_2, \cdots, \sigma_r)^T \in R^{mr}, \qquad \widetilde{\sigma} = \sigma \otimes I_m, \end{cases}$$

where I_m is an $m \times m$ unit matrix, the symbol $A \otimes B$ denotes Kronecker product of the matrices A and B. Then the method (2.5) can be written in more compact form

$$\begin{cases} Y &= h\widetilde{C}_{11} F(Y) + \widetilde{C}_{12} y^{(n-1)}, \\ y^{(n)} &= h\widetilde{C}_{21} F(Y) + \widetilde{C}_{22} y^{(n-1)}, \\ y_n &= \widetilde{\sigma} y^{(n)}. \end{cases} \tag{1.4}$$

Definition 1.1.(cf.[6,7]) *Let k, p, q be real constants with $k > 0$ and $pq < 1$, $G = [g_{ij}]$ a real positive definite symmetric $r \times r$ matrix, $D = diag(d_1, d_2, \cdots, d_s)$ a real nonnegative definite diagonal $s \times s$ matrix, furthermore, for $l > 0$, $\widetilde{D}$ denotes an $l \times l$ nonnegative definite diagonal matrix. The method (1.3) is said to be (k, p, q)-weakly algebraically stable (about the matrices $G, D, \psi(\widetilde{D})$) if the matrix*

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

is nonnegative definite, where

$$M_{11} = kG - C_{22}^T G C_{22} - pC_{12}^T D C_{12} + \phi(\widetilde{D}), \qquad M_{12} = M_{21}^T = C_{12}^T D - C_{22}^T G C_{21} - pC_{12}^T D C_{11},$$
$$M_{22} = C_{11}^T D + DC_{11} - C_{21}^T G C_{21} - pC_{11}^T D C_{11} - qD,$$