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A MODIFIED ALGORITHM OF FINDING AN ELEMENT OF CLARKE GENERALIZED GRADIENT FOR A SMOOTH COMPOSITION OF MAX-TYPE FUNCTIONS*1)

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Abstract

This paper refers to Clarke generalized gradient for a smooth composition of max-type functions of the form: $f(x) = g(x, \max_{j\in J_1} f_{1j}(x), \cdots, \max_{j\in J_m} f_{mj}(x))$, where $x \in \mathbf{R}^n$, J_i, $i = 1, \cdots, m$ are finite index sets, g and $f_{ij}, j \in J_i$, $i = 1, \cdots, m$, are continuously differentiable on $\mathbf{R}^{m+n}$ and $\mathbf{R}^n$, respectively. In a previous paper, we proposed an algorithm of finding an element of Clarke generalized gradient for f, at a point. In that paper, finding an element of Clarke generalized gradient for f, at a point, is implemented by determining the compatibilities of systems of linear inequalities many times. So its computational amount is very expensive. In this paper, we will modify the algorithm to reduce the times that the compatibilities of systems of linear inequalities have to be determined.

Key words: Nonsmooth optimization, Clarke generalized gradient, Max-type function.

1. Introduction

The smooth composition of max-type functions plays an important role in nonsmooth optimization. The general form of this kind of functions is :

$$f(x) = g(x, \max_{j\in J_1} f_{1j}(x), \cdots, \max_{j\in J_m} f_{mj}(x)), \tag{1.1}$$

where $x \in \mathbf{R}^n, J_i, i = 1, \cdots, m$ are finite index sets, g and $f_{ij}, j \in J_i$, $i = 1, \cdots, m$ are continuously differentiable on $\mathbf{R}^{m+n}$ and $\mathbf{R}^n$, respectively. Many publications deal with the problem related to minimizing this class of functions, see for instance [3, 8, 9]. However, authors have to restrict themselves to considering particular cases about f, or take f as a quasidifferentiable function, in the sense of Demyanov and Rubinov [4]. For instance, $g(x, y_1, \cdots, y_m)$, where $x \in \mathbf{R}^n$, is supposed to be nondecreasing with respect to each y_i for $i = 1, \cdots, m$ in [8] and f is taken as a quasidifferentiable function in [3, 9]. Until now, no papers on minimizing f in general case by using the technology of Clarke generalized gradient appear. The present situation may be caused from that one has not found a way to obtain an element of Clarke generalized gradient of f.

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The definition of Clarke generalized gradient for a locally Lipschitzian function and its properties see [1, 2].

Gao and Xia [5] proposed two algorithms of finding an element of Clarke generalized gradient of f, at a point. The algorithms proposed in [5] could be taken as subalgorithms embedded in some algorithms of minimizing f, for instance, bundle methods [7, 10], such that these algorithms become implementable ones. However, the computational amount of the algorithms proposed in [5] is very expensive. One has to determine the compatibilities of systems of linear inequalities $\prod_{i=1}^{m} \mathrm{card} J_i(x)$ times to find an element of Clarke generalized gradient of f, at the point x, where card denotes cardinality, the definition of $J_i(x)$ see (1.2) below. It is this reason why one does apply them to minimizing f immediately.

In this paper, we intend to modify the algorithms proposed in [5] to reduce the computational amount. Now, we go back to [5]. Let $\overline{x} \in \mathbf{R}^n$. Denote

$$J_i(\overline{x}) = \{j \in J_i \mid f_{ij}(\overline{x}) = \max_{j \in J_i} f_{ij}(\overline{x})\}, i = 1, \cdots, m. \tag{1.2}$$

Given each set of indices $j_i \in J_i(\overline{x})$ for $i = 1, \cdots, m$, one construct the following system of linear inequalities

$$\mathrm{L}_{j_1 \cdots j_m} \qquad (\nabla f_{it_i}(\overline{x}) - \nabla f_{ij_i}(\overline{x}))^T y < 0, y \in \mathbf{R}^n, \forall t_i \in J_i(\overline{x}) \setminus \{j_i\}, i = 1, \cdots, m.$$

It is easy to see that $\mathrm{L}_{j_1 \cdots j_m}$ is a system of $\sum_{i=1}^{m} (\mathrm{card} J_i(\overline{x}) - 1)$ strictly linear inequalities with n variables. One has the following two theorems.

Theorem1.1.[Th.2,5] *Suppose there exists a set of indices $j_i \in J_i(\overline{x})$ for $i = 1, \cdots, m$, such that the system of linear inequalities $\mathrm{L}_{j_1 \cdots j_m}$ is consistent. Then $\nabla g(x, f_{1j_1}(x), \cdots, f_{mj_m}(x))|_{x=\overline{x}} \in \partial f(\overline{x})$, where $\partial f(\overline{x})$ refers to Clarke generalized gradient of f, at $\overline{x}$.*

Theorem1.2.[Th.3,5] *Suppose $j, k \in J_i(\overline{x}), j \neq k$ implies $\nabla f_{ij}(x) \neq \nabla f_{ik}(\overline{x})$ for $i = 1, \cdots m$. Then there exists at least one set of indices $j_i \in J_i(\overline{x})$ for $i = 1, \cdots, m$ such that its related system of linear inequalities $\mathrm{L}_{j_1 \cdots j_m}$ is consistent.*

In the light of Theorems 1.1 and 1.2, an algorithm of finding an element of Clarke generalized gradient of f, at $\overline{x}$, is constructed in [5]. The algorithm works under the hypothesis in Theorem 1.2. The general outline of the algorithm is: For each set of indices $j_i \in J_i(\overline{x}), i = 1, \cdots, m$, one determines the compatibility of the system of linear inequalities $\mathrm{L}_{j_1 \cdots j_m}$. If $\mathrm{L}_{j_1 \cdots j_m}$ is consistent, calculate $\xi = \nabla g(x, f_{1j_1}(x), \cdots, f_{mj_m}(x))|_{x=\overline{x}}$, which is an element of Clarke generalized gradient for f, at $\overline{x}$.

For dealing with more general case in which hypothesis in Theorem 1.2 may be not satisfied, a modified algorithm that works without the hypothesis: $\nabla f_{ij}(\overline{x}) \neq \nabla f_{ik}(\overline{x}), \forall j, k \in J_i(\overline{x}), j \neq k, i = 1, \cdots, m$, is presented. The general outline of the modified algorithm is: Determine index sets $\overline{J}_i(\overline{x})$ for $i = 1, \cdots, m$ according to the rule below:

$$\overline{J}_i(\overline{x}) \subset J_i(\overline{x}), \quad i = 1, \cdots, m,$$

$$\nabla f_{ij}(\overline{x}) \neq \nabla f_{ik}(\overline{x}), \forall j, k \in \overline{J}_i(\overline{x}), j \neq k, i = 1, \cdots, m,$$

$$\forall\ t_i \in J_i(\overline{x}), \exists\ k_i \in \overline{J}_i(\overline{x}) \ \text{ such that } \nabla f_{it_i}(\overline{x}) = \nabla f_{ik_i}(\overline{x}), i = 1, \cdots, m.$$