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ASYMPTOTIC ANALYSIS OF SHELLS VIA Γ-CONVERGENCE*[1)]

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Abstract

We give a new justification of the linear membrane and flexural shell models. We prove that the sequence of scaled energy functionals associated with the scaled problem Γ-converges to the energy functional associated with a two-dimensional model. This two-dimensional model is a membrane or flexural one, depending on the geometric and kinematic conditions. Then, a classical argument allows to give a new proof of the convergence theorems recently obtained by P.G. Ciarlet, V. Lods and B. Miara.

Key words: Γ-converges, linear elastic shell

Introduction

The deformations of an elastic body submitted to forces are governed by three-dimensional mathematical equations. This means that the unknown, which is the vector formed by the components of the displacement, depends on three variables. however, when the elastic body is "thin" in one dimension, for instance when it is a shell, one can use two-dimensional shell models, such as those of Naghdi, Koiter, Budiansky-Sanders etc. Thus, the important point is to explain which model is the "good" one in a given situation and why. Hence, an important aspect of the mathematical analysis in elasticity consists in studying the validity of the two-dimensional equations to describe the physical behavior of a three-dimensional body. This is what is called the justification of the model.

Deriving lower-dimensional models can be achieved through a formal asymptotic analysis. The method is the following: first, one has to make the "scalings" on the unknown and the "right" assumptions on the forces in order to set the problem over a fixed domain, i.e, a domain independent of the thickness ε. Next, it is assumed that the scaled three-dimensional displacement field obtained in this fashion can be expanded in powers of the small parameter ε. Finally, replacing this formal expansion in the variational equations, one can identify the leadin term by equating to 0 the coefficients of the powers of ε.

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In the study of linearly elastic shells, the first contribution of that kind is due to [1]. Then, [2] pointed out the importance of the geometry of the shell: depending on the geometric and kinematic conditions, the formal asymptotic analysis leads to identify one of two distinct models: the "membrane" model or the "flexural" model. Thus, it is not possible to derive these two models simultaneously for shells, unlike the case of plates. For other works in this spirit, see [3–6].

Essentially, a two-dimensional model is considered justified when one can prove convergence of the three-dimensional unknown to the leading term of the asymptotic expansion, as the thickness ε of the shell goes to zero. In the linear case, the articles of [7, 8], [9] give the complete justification of the membrane and flexural models by using the techniques of asymptotic analysis. For nonlinear membranes, such results were obtained by [10], using Γ-convergence and following the approach of [11].

Here, we give another method to obtain convergence theorems in the linear case using Γ-convergence theory. A similar approach was done for linearly elastic plates by [12].

We study separately the membrane case and the flexural case. First, we recall the main notations about the geometry of the shell, and we make appropriate scalings, in order to define the scaled three-dimensional problem. Next, we prove the Γ-convergence of the energy functionals associated with the scaled three-dimensional problem to a functional corresponding to a variational problem posed over a two-dimensional domain. We then deduce the weak convergence of the displacements, the strong convergence being obtained as in [7], [9].

1. The Three-Dimensional Shell Problem in Linearized Elasticity

We begin with geometric preliminaries. Throughout this work, Greek indices and exponents (except ε) belong to the set $\{1,2\}$, Latin indices and exponents (except when used to index sequences) take their values in the set $\{1, 2, 3\}$, and we use the summation convention on repeated indices and exponents.

Let ω be a bounded, open and connected subset of $\boldsymbol{R}^2$, with a Lipschitz-continuous boundary γ. We note $y = (y_\alpha)$ a generic point of $\bar{\omega}$, and $\partial_\alpha := \partial/\partial y_\alpha$ the partial derivatives. let $\varphi : \bar{\omega} \to \boldsymbol{R}^3$ be an injective mapping, at least of class $\mathcal{C}^3$. We assume that the two vectors

$$\boldsymbol{a}_\alpha(y) := \partial_\alpha \varphi(y)$$

are linearly independent at all points $y \in \bar{\omega}$. They form the *covariant basis* of the tangent plane to the surface $S = \varphi(\bar{\omega})$ at the point $\varphi(y)$; the two vectors $\boldsymbol{a}^\alpha(y))$ defined by

$$\boldsymbol{a}^\alpha(y) \cdot \boldsymbol{\alpha}_\beta(y) = \delta^\alpha_\beta$$

constitute the contravariant basis at this same point $\varphi(y)$. We also define the vector

$$\boldsymbol{a}_3 = \boldsymbol{a}^3 := \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|}.$$