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THE NUMERICAL STABILITY OF THE θ-METHOD FOR DELAY DIFFERENTIAL EQUATIONS WITH MANY VARIABLE DELAYS*

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Abstract

This paper deals with the asymptotic stability of theoretical solutions and numerical methods for the delay differential equations(DDEs)

$$\begin{cases} y'(t) & = ay(t) + \sum_{j=1}^{m} b_j y(\lambda_j t) \quad t \geq 0, \\ y(0) & = y_0, \end{cases}$$

where $a, b_1, b_2, \ldots, b_m$ and $y_0 \in C$, $0 < \lambda_m \leq \lambda_{m-1} \leq \ldots \leq \lambda_1 < 1$. A sufficient condition such that the differential equations are asymptotically stable is derived.And it is shown that the linear θ-method is ΛGP_m-stable if and only if $\frac{1}{2} \leq \theta \leq 1$.

Key words: Delay differential equation, Variable delays, Numerical stability, θ-methods.

1. Introduction

In this paper,we will investigate the numerical solutions of the following initial value problems for DDEs with many variable delays

$$\begin{cases} y'(t) & = ay(t) + \sum_{j=1}^{m} b_j y(\lambda_j t) \quad t \geq 0, \\ y(0) & = y_0, \end{cases} \tag{1.1}$$

where $a, b_1, b_2, \ldots, b_m$ and $y_0 \in C$, $0 < \lambda_m \leq \lambda_{m-1} \leq \ldots \leq \lambda_1 < 1$.It is difficult to investigate numerically the long time dynamical behaviour of the exact solution due to limited computer memory. To avoid this problem we transform (1.1) into the differential

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equations with constant time lags in the following way.(see [3]) Let $x(t) = y(e^t)$ for $t \geq \log \lambda_m$.Then $x(t)$ satisfies the following initial value problems

$$\begin{cases} x'(t) & = ae^t x(t) + \sum_{j=1}^{m} b_j e^t x(t + \log \lambda_j) \quad t \geq 0, \\ x(t) & = y(e^t) := \Phi(t) \qquad t \in [\log \lambda_m, 0], \end{cases} \tag{1.2}$$

where $y(t), 0 \leq t \leq e^0 = 1$,can be obtained numerically by using θ -method to (1.1). Then,let us consider the following linear test equations which were introduced in [4],

$$\begin{cases} y'(t) & = a(t)y(t) + b(t)y(t-\tau) \quad \tau > 0, t \geq 0, \\ y(t) & = \Phi(t) \quad -\tau \leq t \leq 0, \end{cases} \tag{1.3}$$

where $y : [-\tau, +\infty) \to C, \quad a, b : [0, +\infty) \to C$.

If a(t) and b(t) are continuous and satisfy

$$Re(a(t)) \leq -\beta < 0, \tag{1.4a}$$

$$|b(t)| \leq -q \cdot Re(a(t)), 0 \leq q < 1 \tag{1.4b}$$

and $\Phi(t)$ is continuous,then the solution $y(t)$ of (1.3) is asymptotically stable, namely, $y(t) \to 0$,as $t \to \infty$.

In [4],the authors introduced two definitions of stability based on the test equations (1.3) as follows.

Definition 1. *A numerical method for DDEs is called TP-stable if, under the condition (1.4),the numerical solution y_n of (1.3) satisfies*

$$\lim_{n \to \infty} y_n = 0 \tag{1.5}$$

for every stepsize h such that $h = \tau / l$ where $l \geq 1$ is a positive integer.

Definition 2. *A numerical method for DDEs is called TGP-stable if, under the condition (1.4),the numerical solution y_n of (1.3) satisfies (1.5) for every stepsize $h > 0$.*

It is the purpose of this paper to investigate the asymptotic stability behaviour of the theoretical solution and the numerical solution of (1.1).In Section 2,we derive a sufficient condition for (1.1) such that the solution of (1.1) is asymptotically stable. In Section 3,it is proven that the linear θ-method is ΛGP_m-stable if and only if $\frac{1}{2} \leq \theta \leq 1$.

2. Asymptotic Stability Of The Theoretic Solution Of DDEs

Now we consider the following equations:

$$\begin{cases} x'(t) & = a(t)x(t) + b_2(t)x(t-\tau_2) + b_1(t)x(t-\tau_1) \quad t \geq 0, \tau_2 \geq \tau_1 > 0, \\ x(t) & = \Phi(t) \qquad t \leq 0, \end{cases} \tag{2.1}$$

where $x : R \to C, \quad a, b_1, b_2 : [0, +\infty) \to C$,and $\Phi : (-\infty, 0] \to C$.