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# REAL-VALUED PERIODIC WAVELETS: CONSTRUCTION AND RELATION WITH FOURIER SERIES*[1)]

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**Abstract**

In this paper, we construct the real-valued periodic orthogonal wavelets. The method presented here is new. The decomposition and reconstruction formulas involve only 4 terms respectively. It demonstrates that the formulas are simpler than that in other kinds of periodic wavelets. Our wavelets are useful in applications since it is real valued. The relation between the periodic wavelets and the Fourier series is also discussed.

*Key words*: Periodic wavelet, Multiresolution, Fourier series, Linear independence.

## 1. Introduction

Wavelets have recently received a great deal of attention in such areas as signal processing and image processing ([12], [8]). Various methods to construct wavelets have been given ([14], [13], [9], [7]). It is well known that in mathematics and mathematic physics many periodic problems are encountered. In application areas, the input signals are usually finite length which may lead extra computations. To avoid this, various efforts have been made ([5], [10], [21]), among which periodization method is an important approach, i.e., the finite length input signal is first periodized, then a periodic wavelet is used which motivated an extensive study of periodic wavelets.

Y. Meyer ([14]) studied periodic multiresolutions by periodizing known wavelets. Perrier and Basdevant ([16]) stated the construction and algorithm of periodic wavelets, their algorithm makes heavy use of the fast Fourier transform(FFT). Chui and Mhasker[6]

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constructed the trigonometric wavelets. Plonka and Tasche ([17], [18]) studied $p$-periodic wavelets for general periodic scaling functions. Their algorithms ([19]) are based on Fourier technique. Chen Han-Lin made a full study of periodic wavelets when the scaling functions are derived from different kinds of spline functions (see [1], [2], [3], [4]). Each equation in the decomposition and reconstruction algorithms involves only two terms which does not depend on the regularity of the underlying wavelets. The discret Fourier transform is used implicitly. The approximation error estimations are also given. Koh, Lee and Tan ([11]) gave a general framework of periodic wavelets where two terms are obtained and the two-term algorithms operate on the frequency domain is also realized. Narcowich and Ward[15] investigated the periodic scaling functions and wavelets generated by continuously differentiable periodic functions with positive Fourier coefficients. They also discussed the localization of scaling functions and wavelets. The method of using the periodic wavelets, e.g., to denoise and to detect singularity, is also pointed out.

Our interest in this paper is to construct real-valued periodic orthogonal wavelets. The relation between the periodic wavelets and the Fourier series is also discussed. Our method to construct periodic wavelet is quite different from Narcowich and Ward's ([15]). The conditions of the underlying function $\varphi$ is original.

This chapter is organized as follows. We will finish this section with some notations. The periodic scaling functions and nested subspaces will be constructed in Section 1. In Section 2, the dilation equations and periodic wavelets are discussed. Section 3 will devoted to the discussion of the relations between periodic wavelets and the Fourier series. Some examples will be given in Section 4.

We will use the following notations.

Let $T = Kh$ where $K$ is a positive even integer, $h$ a positive real number,$K = 2N$. We also use $N_j := 2^j N, K_j := 2^j K, h_j := T/K_j = h/2^j$. Note that $h_j K_j = hK = T$. $\overset{o}{L}_2\ [0,T]$ represents the set of all periodic, square-summable functions defined on $[0,T]$, equipped with the inner product $< f,g >= \dfrac{1}{T}\int_0^T f(x)\overline{g(x)}dx$.

## 2. The Scaling Functions

In this this section, we will construct the scaling functions and discuss their properties. To do this, we suppose that a compactly supported real valued function $\varphi(x) \in L^2(\mathcal{R})$ satisfies

(i) For some $p \in \mathcal{Z}^+, 2p \leq N$ the support of $\varphi : supp\varphi \subset [-ph, ph]$

(ii) $\varphi$ is refinable, i.e. there exists $\{c_k\} \in l^2$, s.t.

$$\varphi(x) = \sum_{k\in Z} c_k \varphi(2x - kh) \tag{2.1}$$