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KAHAN'S INTEGRATOR AND THE FREE RIGID BODY DYNAMICS*

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Abstract

Kahan's integrator for the free rigid body dynamics is described and some of its properties are pointed out.

Key words: Kahan's integrator, Free rigid body.

1. Introduction

Among other unconventional numerical methods, Kahan[1] has suggested a discretization of a simple Lotka-Voltera system with the property that the computed points do not spiral. The motivation of this behaviour was given recently by Sanz-Serna[3] by showing that Kahan's integrator is a symplectic one.

The goal of our paper is to study some properties of this integrator in the particular case of the free rigid body.

2. The Free Rigid Body

Let $SO(3)$ be the Lie group of all 3×3 orthogonal matrices with determinant one and $so(3)$ its Lie algebra. It can be canonically identified with $\mathbf{R}^3$ via the map "$\wedge$" given by:

$$\wedge : \begin{bmatrix} 0 & -m_3 & m_1 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{bmatrix} \in so(3) \mapsto \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} \in \mathbf{R}^3.$$

Moreover, "$\wedge$" is a Lie algebra isomorphism between $(so(3), [\cdot,\cdot])$ and $(\mathbf{R}^3, \times)$.

Then the Euler angular momentum equations of the free rigid body can be written on $(so(3))^* \simeq (\mathbf{R}^3)^* \simeq \mathbf{R}^3$ in the following form:

$$\begin{cases} \dot{m}_1 = a_1 m_2 m_3 \\ \dot{m}_2 = a_2 m_1 m_3 \\ \dot{m}_3 = a_3 m_1 m_2 \end{cases} \tag{2.1}$$

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where

$$a_1 = \frac{1}{I_3} - \frac{1}{I_2};\ a_2 = \frac{1}{I_1} - \frac{1}{I_3};\ a_3 = \frac{1}{I_2} - \frac{1}{I_1},$$

I_1, I_2, I_3 being the components of the inertia tensor and we suppose as usually that

$$I_1 > I_2 > I_3.$$

It is not hard to see that the system (2.1) is an Hamilton-Lie-Poisson system with the phase space $\mathbf{R}^3$, the Poisson bracket given by the matrix:

$$\Pi = \begin{bmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{bmatrix}, \tag{2.2}$$

which is in fact the minus-Lie-Poisson structure on $so(3)^*$ and the Hamiltonian H given by

$$H = \frac{1}{2}\left[\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3}\right]. \tag{2.3}$$

Moreover, a Casimir of our configuration $(\mathbf{R}^3, \Pi)$ is given by

$$C = \frac{1}{2}[m_1^2 + m_2^2 + m_3^2], \tag{2.4}$$

i.e. that the coadjoint orbits of $(so(3))^* \simeq \mathbf{R}^3$ are concentric spheres.

Since H and C are constants of motion it follows that the dynamics takes place at the intersection of the ellipsoid

$$H = \text{constant}$$

with the sphere

$$C = \text{constant}.$$

3. Kahan's Integrator

For the free rigid body equations (2.1), Kahan's integrator can be written in the following form:

$$\begin{cases} m_1^{n+1} - m_1^n = \dfrac{ha_1}{2}(m_2^{n+1}m_3^n + m_3^{n+1}m_2^n) \\ m_2^{n+1} - m_2^n = \dfrac{ha_2}{2}(m_1^{n+1}m_3^n + m_3^{n+1}m_1^n) \\ m_3^{n+1} - m_3^n = \dfrac{ha_3}{2}(m_2^{n+1}m_1^n + m_1^{n+1}m_2^n) \end{cases} \tag{3.1}$$

Now a long but straightforward computation or using eventually MAPLE V leads us to:

Theorem 3.1. *The following statements are equivalent:*

(i) *Kahan's integrator is a Poisson integrator;*

(ii) *Kahan's integrator is energy preserving.*