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NUMERICAL COMPUTATION OF BOUNDED SOLUTIONS FOR A SEMILINEAR ELLIPTIC EQUATION ON AN INFINITE STRIP*

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Abstract

In this paper, we consider the computation of bounded solutions of a semilinear elliptic equation on an infinite strip. The dynamical system approach and reduction on center manifold are used to overcome the difficulties in numerical procedure.

Key words: Numerical method, Nonlinear PDE, Center manifold.

1. Introduction

Consider elliptic problem on an infinite strip of R^2

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(\lambda, y, u) + g(\lambda, \epsilon, x, y) = 0 \tag{1.1}$$

where $(x, y) \in (-\infty, \infty) \times (0, 1)$ and f, g are smooth functions of thier arguments, λ, ϵ two real parameters. We are interested in the bounded solutions of (1.1) with the following conditions

$$u(x, 0) = u(x, 1) = 0, \quad x \in R \tag{1.2}$$

and

$$\lim_{x \to -\infty} u(x, y) = 0, \quad y \in (0, 1). \tag{1.3}$$

Some problems arising in applied mathematics are given by the formulation (1.1) with conditions (1.2) and (1.3), for example, the description of the steady flow of an inviscid nondiffusive fluid through a channel of varying depth(see A. Meilke[7]). Here we concern the numerical computation of the bounded solutions of (1.1) with (1.2) and (1.3), i.e., that solution satisfying

$$\sup_{x,y} |u(x, y)| < +\infty. \tag{1.4}$$

To do this, we shall meet some difficulties from two aspects, unboundedness of domain and nonlinearity of function f. In order to overcome the difficulty from unboundedness of domain, the boundary conditions at an artificial boundary are often used and then the boundary-value problems on the finite domain are solved (see T.Hagstrom and H.B.Keller [2] and its references). However, the multi-solution of our problem which

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is from the nonliearity makes it difficult to compute numerically even though artificial boundary conditions are used.

We propose here another approach to compute the solutions of problem (1.1). The first step of our approach is to transform (1.1) with boundary condition (1.2) into infinite dimensional formally dynamical system which is follows the idea of K.Kirschgässner[3], A.Mielke[7] and Ma[5]. Then, the bounded solutions of (1.1) will be found as the special orbits–homoclinic or heteroclinic or half-periodic orbits of the formally dynamical system. The second step of our approach is to study numerically the planar dynamical system reduced from infinite dimentional system by use of center manifold theory. The porpuse of this step is to provid good prediction of special orbits of system obtained by the first step. Finally, we calculate numerically the special solutions of the formally dynamical system. To this end, of course, it is neccessary to approximate the infinite dimensional system by a finite dimensional one and to give a artificial boundary condition. We use the semi-discrezation on y and the projection boundary conditions. Meanwhile, we also use a predict-correct procedure with an initial prediction which is constructed by use of the results in step two.

The outline of this paper is as follows: in Sec. 2, we describe the procedures to transform (1.1) and (1.2) into the infinite dimensional system and to reduce it into a planar dynamical system by use of the center manifold theory. In Sec. 3, we give a numerical study of reduced system. In Sec.4, the predict-correct procedure to solve problem (1.1)–(1.4) is described. In last section, a numerical implementation of our approach is given by a example.

2. Formally Dynamical System Formulation of Problem

Following Kirschgässner[3] and Mielke[7], we now transform our problem (1.1)–(1.2) into an infinite dimensional system. Assume that $f(\lambda, y, 0) \equiv 0$ and define the linear operator $L(\lambda)$ in $L^2(0,1)$ as

$$L(\lambda)\phi := -\frac{\partial^2\phi}{\partial y^2} - \frac{\partial f}{\partial u}(\lambda, y, 0)\phi,$$
$$\forall \phi \in D(L(\lambda)) = H_0^1(0,1) \bigcap H^2(0,1).$$

Then (1.1) with (1.2) can be understood as a nonlinear differential equation

$$\frac{d^2u}{dx^2} - L(\lambda)u + \bar{f}(\lambda, u) + \bar{g}(\lambda, \epsilon, x) = 0, \tag{2.1}$$

where $u : (-\infty, \infty) \to L^2(0,1)$, $\bar{f} : \Lambda \times \mathrm{L}^2(0,1) \to L^2(0,1)$, $\bar{g} : \Lambda \times (-\epsilon_0, \epsilon_0) \times (-\infty, \infty) \to L^2(0,1)$ are defined by

$$\begin{aligned} u(x)(y) =& u(x,y), \\ \bar{f}(\lambda, u)(y) =& f(\lambda, y, u) - \frac{\partial f}{\partial u}(\lambda, y, 0)u, \\ \bar{g}(\lambda, \epsilon, x)(y) =& g(\lambda, \epsilon, x, y) \end{aligned}$$