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THE BOUNDARY INTEGRO-DIFFERENTIAL EQUATIONS OF A BIHARMONIC BOUNDARY VALUE PROBLEM*1)

Hou-de Han
(*Departmentof Applied Mathematics, Tsinghua University, Beijing 100084, China*)
Wei-jun Tang
(*Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics, Beijing 100088, China*)

Abstract

In this paper, a new method of boundary reduction is proposed, which reduces the biharmonic boundary value problem to a system of integro-differential equations on the boundary and preserves the self-adjointness of the original problem. Moreover, a boundary finite element method based on this integro-differential equations is presented and the error estimates of the numerical approximations are given. The numerical examples show that this new method is effective.

Key words: Boundary Iintegro-differential equations, Bihamonic boundary value problem

1. Introduction

We consider a homogeneous isotropic and linear elastic Kirchhoff plate under lateral load distributed over the plate $\Omega \times [-\frac{h}{2}, \frac{h}{2}]$. The domain $\Omega \in R^2$ is bounded with the smooth boundary Γ. In the static equilibrium, we consider the free type boundary condition on Γ. Then the deflection u satisfies the following problem:

$$\begin{cases} \Delta^2 u = \dfrac{q}{D}, & \text{in } \Omega, \\ M(x, n_x)u = 0, & \text{on } \Gamma, \\ T(x, n_x)u = 0, & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where $D = \dfrac{E_0 h^3}{12(1-\nu^2)}$, is the bending stiffness of the plate with h being the plate thickness and E_0 and $\nu (0 < \nu < \frac{1}{2})$ being the modulus and Poisson's ratio respectively, q denotes the lateral loading; the boundary differential operators $M(x, n_x)$, $T(x, n_x)$ are given by:

$$\begin{aligned} M_x \equiv & M(x, n_x) = \nu \Delta_x \\ & + (1-\nu)\Big[n_1^2(x)\frac{\partial^2}{\partial x_1^2} + n_2^2(x)\frac{\partial^2}{\partial x_2^2} + 2n_1(x)n_2(x)\frac{\partial^2}{\partial x_1 \partial x_2}\Big], \end{aligned} \tag{1.2}$$

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$$T_x \equiv T(x, n_x) = -\frac{\partial \Delta_x}{\partial n_x} + (1-\nu)\frac{\partial}{\partial s_x}[n_1(x)n_2(x)\Big(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\Big) - ((n_1(x))^2 - (n_2(x))^2)\frac{\partial^2}{\partial x_1 \partial x_2}], \tag{1.3}$$

where $n_x = (n_1(x), n_2(x))^T$ denotes the unit outer normal vector at $x \in \Gamma$ and $s_x = (-n_2(x), n_1(x))^T$ is the unit tangential vector at $x \in \Gamma$. For convenience, from now on we suppose that the bending stiffness D has been normalized to $D = 1$. Because the lateral loading $q(x)$ in (1.1) can always be eliminated by substracting a volume potential, hence the problem (1.1) can be reduced to the following problem:

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ M_x u = m & \text{on } \Gamma, \\ T_x u = t & \text{on } \Gamma, \end{cases} \tag{1.4}$$

for given functions $m(x), t(x)$ on the boundary Γ. Let $\Omega^c = R^2 \backslash \Omega$, then we also consider the boundary value problem on the unbounded domain Ω^c:

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega^c, \\ M_x u = m & \text{on } \Gamma, \\ T_x u = t & \text{on } \Gamma, \\ & u(x) \text{ satisfies the linear - logarithmic growth condition} \\ & \text{(see [11], p468. (8.165)), when } |x| \to \infty. \end{cases} \tag{1.5}$$

The operators M_x and T_x can be rewritten in the following form:

$$M_x = \Delta_x - (1-\nu)\frac{\partial^2}{\partial s_x^2} - (1-\nu)\omega(x, n_x)\frac{\partial}{\partial n_x}, \tag{1.6}$$

$$T_x = -\frac{\partial \Delta_x}{\partial n_x} - (1-\nu)\frac{\partial^3}{\partial s_x^2 \partial n_x} + (1-\nu)\frac{\partial}{\partial s_x}\Big[\omega(x, n_x)\frac{\partial}{\partial s_x}\Big], \tag{1.7}$$

where $\omega(x, n_x) = n_1(x)\dfrac{dn_2(x)}{ds_x} - n_2(x)\dfrac{dn_1(x)}{ds_x}$.

We will reduce the problem (1.4) to a system of boundary integro-differential equations by an indirect method.

Let

$$u(x) = \int_\Gamma M_y E(x, y) f_1(y) ds_y + \int_\Gamma T_y E(x, y) f_2(y) ds_y + p_1(x), \ x \in \Omega, \tag{1.8}$$

be the solution of problem (1.4). Here $p_1(x)$ is an arbitrary polynomial of degree one, $E(x, y) = \dfrac{1}{8\pi} r^2 \log r$, with $r = |x - y|$ is a fundamental solution of biharmonic equation, f_1, f_2 are two unknown density functions.

For any $x \notin \Gamma$, and an arbitrary unit vector n_x, we have

$$M_x u(x) = \int_\Gamma M_x M_y E(x, y) f_1(y) ds_y + \int_\Gamma M_x T_y E(x, y) f_2(y) ds_y, \ x \notin \Gamma, \tag{1.9}$$

$$T_x u(x) = \int_\Gamma T_x M_y E(x, y) f_1(y) ds_y + \int_\Gamma T_x T_y E(x, y) f_2(y) ds_y, \ x \notin \Gamma. \tag{1.10}$$