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ON MATRIX UNITARILY INVARIANT NORM CONDITION NUMBER*

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Abstract

In this paper, the unitarily invariant norm $\|\cdot\|$ on $\mathbb{C}^{m\times n}$ is used. We first discuss the problem under what case, a rectangular matrix A has minimum condition number $K(A) = \|A\| \, \|A^+\|$, where A^+ designates the Moore-Penrose inverse of A; and under what condition, a square matrix A has minimum condition number for its eigenproblem? Then we consider the second problem, i.e., optimum of $K(A) = \|A\| \, \|A^{-1}\|_2$ in error estimation.

Key words: Matrix, unitarily invariant norm, condition number

1. Introduction

Since 1984, several chinese mathematicians have obtained many results bout matrix operator norm condition number[11,12,18].

Two kinds matrix condition numbers [9] are :

(1) If $A \in \mathbb{C}^{n\times n}$ is nonsingular, the number $K_\alpha(A) = \|A\|_\alpha \|A^{-1}\|_\alpha$ is called the α-norm condition number of A for its inverse, where $\|\cdot\|_\alpha$ is some matrix norm, such as the 2-norm, *Hölder*-norm, F-norm, etc..

Furthermore, we can generalize the inverse condition number to rectangular matrix case [1], [8], $K(A) = \|A\|_\alpha \|A^+\|_\beta$, and allows $\alpha \neq \beta$.

(2) For a square matrix $A \in \mathbb{C}^{n\times n}$, set

$$V_A = \{X \mid X \in \mathbb{C}^{n\times n},\ X^{-1}AX = J_A, \text{a Jordan form of } A\}. \tag{1.1}$$

Then the number

$$J_\alpha = \inf_{X\in V_A} \{\|X\|_\alpha \|X^{-1}\|_\alpha\} \tag{1.2}$$

is called the α-norm condition number of A for its eigenproblem.

Wilkinson[9] pointed out that a) If matrix A is normal, then $J_2(A) = 1$. b) If A is unitary, then $K_2(A) = 1$.

Zheng[11,12] obtained the necessary and sufficient conditions for minimizing two kinds of p-norm condition numbers $(1 \le p \le \infty)$.

Zheng and Zhao[8] obtained the structures of p-norm isometric matrix $A \in \mathbb{C}^{m\times n}$ and the bounds of $K_p(A) = \|A\|_p \|A^+\|_p$ $(1 \le p \le \infty)$; Wang and Chen obtained the structures of a rectangular matrix A with minimum p-norm condition number $(1 \le p \le \infty,\ p \neq 2)$.

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All the above results are concerned with matrix operator norms.

Other results associated with matrix operator norm condition number are given by Yang[10], i.e., the optimum of $K(A) = \|A\|\ \|A^{-1}\|$ in the error estimation of linear equation $Ax = b$ and the process of computing A^{-1}.

In this paper, another important kind matrix norm, the unitarily invariant norm on $\mathbb{C}^{m\times n}$ (UIN) is discussed, and some results associated condition number are obtained.

The rest of the paper is arranged as follows. Section 2 is preliminary. In Section 3, the structures of the rectangular matrices with minimum UIN condition number $K(A) = \|A\|\ \|A^{+}\|$ are discussed. In Section 4, the condition for a square matrix A possesses minimum UIN condition number for its eigenproblem is obtained. Finally, Section 5 is used to describe some results about the optimum of $K(A) = \|A\|\ \|A^{-1}\|_2$ in error estimation, where $\|\cdot\|$ designates a UIN.

2. Preliminaries

Definition 2.1[6,7]**.** *A norm $\|\cdot\|$: $\mathbb{C}^{n\times n} \to \mathbb{R}$ is called unitarily invariant (UIN) if it satisfies :*

(1) $\|UAV\| = \|A\|$, $\forall A, U, V \in \mathbb{C}^{n\times n}$, *and* $U^H U = V^H V = I_n$.

(2) $\|A\| = \|A\|_2$ *if* $rank(A) = 1$.

Definition 2.2[6,7]**.** *A norm $\Phi : \mathbb{R}^n \to \mathbb{R}$ is called a symmetric guage function (SG) if it satisfies :*

(1) *For any permutation matrix P, $\Phi(Px) = \Phi(x)$, $\forall x \in \mathbb{R}^n$.*

(2) $\Phi(|x|) = \Phi(x)$, *where* $x = (\xi_1, \cdots, \xi_n)^T$, *and* $|x| = (|\xi_1|, \cdots, |\xi_n|)^T$.

(3) $\Phi(e_1) = 1$, *where e_1 is the first column of I_n.*

The conception of unitarily invariant norm can be generalized to the rectangular matrix case [6], [7, p. 79], and many properties of the UIN can be found in [6] [7] etc..

Lemma 2.1. *Let $\Phi_p : \mathbb{R}^m \to \mathbb{R}$ be a function defined by*

$$\Phi_p(x) = \|x\|_p = \Big(\sum_{i=1}^{m} |\xi_i|^p\Big)^{1/p},\ (1 \le p \le \infty). \tag{2.1}$$

Then Φ_p is a SG on $\mathbb{R}^m$.

Proof. It is obvious that Φ is the *Hölder* norm on $\mathbb{R}^m$ [5], and satisfies (1) (2) (3) of Definition 2.2. □

If $A \in \mathbb{C}^{k\times l}$, Φ is a SG on $\mathbb{R}^n$, $m = \min\{k.l\} \le n$, $\sigma_1, \cdots, \sigma_m$ are the singular values of A. Then a UIN on $\mathbb{C}^{k\times l}$ may be defined by [6, p. 79]

$$\|A\|_\Phi = \Phi(\sigma_1, \cdots, \sigma_m, 0 \cdots, 0). \tag{2.2}$$

It is easy to see that[6] $\|A\|_{\Phi_0} = \|A\|_2,\ and\ \|A\|_{\Phi_2} = \|A\|_F$.

Definition 2.3. *If Φ_p is defined by (2.1), $\|\cdot\|_\Phi$ is defined by (2.2). Then $\|\cdot\|_{\Phi_p}$ is called a pUIN on $\mathbb{C}^{k\times l}$.*

Lemma 2.2. *Suppose $0 \ne A \in \mathbb{C}^{m\times n}$, $\|\cdot\|$ is a UIN family. Then*

$$K(A) = \|A\|\ \|A^{+}\|_2 \ge 1,\ and\ K(cA) = K(A)\ when\ c \ne 0. \tag{2.3}$$