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NUMERICAL ANALYSIS FOR A MEAN-FIELD EQUATION FOR THE ISING MODEL WITH GLAUBER DYNAMICS*[1)]

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Abstract

In this paper, a mean-field equation of motion which is derived by Penrose (1991) for the dynamic Ising model with Glauber dynamics is considered. Various finite difference schemes such as explicit Euler scheme, predictor-corrector scheme and some implicit schemes are given and their convergence, stabilities and dynamical properties are discussed. Moreover, a Lyapunov functional for the discrete semigroup $\{S\}_{n>0}$ is constructed. Finally, numerical examples are computed and analyzed. it shows that the model is a better approximation to Cahn-Allen equation which is mentioned in Penrose (1991).

1. Introduction

We consider the following mean-field equation of motion for the dynamic Ising model on a periodic lattice Λ:

$$\begin{cases} \mathbf{u}_t + \mathbf{u} = \tanh(\beta \mathbf{A}\mathbf{u}) \quad t > 0 & (1.1\text{a}) \\ \mathbf{u}(0) = \mathbf{u}_0 \in V_\Lambda & (1.1\text{b}) \\ \mathbf{u}_{a+N\mathbf{e}^i} = \mathbf{u}_a \qquad a \in \Lambda,\ 1 \le i \le d & (1.1\text{c}) \end{cases}$$

where Λ denotes the lattice of $\mathbf{Z}^d$ with N^d sites defined by $\Lambda := \Big\{a : a = \sum_{i=1}^{d} a_i \mathbf{e}^i$, $a_i \in \mathbf{Z},\ 1 \le a_i \le N\Big\}$ with $\{\mathbf{e}^i\}$ being the standard unit vectors of $\mathbf{Z}^d$. We say that Λ is a d-dimensional lattice. We denote by V_Λ the N^d dimensional space of lattice vectors $\mathbf{v} = (v_a)_{a\in\Lambda^*}$ satisfying $v_{a+N\mathbf{e}^i} = v_a$. Here $\mathbf{u} = (u_a)_{a\in\Lambda}$ and u_a denotes the expectation of the spin at site a of the lattice and Λ^* is defined by $\Big\{a : a = \sum_{i=1}^{d} a_i \mathbf{e}^i,\ a_i \in Z\Big\}$.

The $N^d \times N^d$ symmetric matrix $\mathbf{A}$ is defined by, for $v \in V_\Lambda$

$$\{\mathbf{A}\mathbf{v}\}_a := \sum_{b\in\Lambda} E_{ab} v_b \tag{1.2}$$

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where $J_{ab} = JE_{ab}$ ($J > 0$) is the Ising interaction between sites a and b satisfying, for all $a, b \in \Lambda$

$$(i). \quad E_{ab} \geq 0 \qquad (ii). \quad E_{ab} > 0 \Longleftrightarrow b \in N(a) \qquad (iii). \quad E_{ab} \leq 1. \tag{1.3}$$

Here $N(a)$ denotes the neighborhood of the site a defined by $N(a) = \Big\{b : \sum_{i=1}^{d} |a_i - b_i| = 1\Big\}$. The parameter $\beta = J/\theta$, where $\theta(> 0)$ is the absolute temperature. Furthermore throughout the paper we use the convention that for any lattice vector $\mathbf{u}$, the component at site a in $(\mathbf{u})_a = u_a$ and for any $f : \mathbf{R} \to \mathbf{R}$, $\{f(\mathbf{u})\}_a = f(u_a)$. The dynamical system (1.1) was derived by Penrose[1] from an Ising model on the lattice Λ. It approximately represents the behavior in the mean of the Ising model with Glauber (spin-flip) stochastic dynamics, Glauber[2]. Existence and bounded of absorbing sets, global attractor for (1.1) are studied by Lu Bainian[3], and the bifurcation solutions for the steady-state equation of the equation (1.1) also are discussed in [3].

In this paper we shall construct some explicit and implicit finite difference approximations and their convergence, stability, dynimical properties and long time behavior for the equation (1.1).

For simplicity, we shall use the same notations and abbreviations as used in [3]

$$\theta_a := J \sum_{b \in N(a)} E_{ab} \tag{1.4a}$$

$$\theta_c := \max_{a \in \Lambda} \theta_a = J \|\mathbf{A}\|_\infty, \tag{1.4b}$$

where $\|\mathbf{A}\|_\infty$ is the infinity norm of the matrix $\mathbf{A}$ and given by $\|\mathbf{A}\|_\infty := \max_{a \in \Lambda} \sum_{b \in N(a)} E_{ab}$.

The discrete weighted L^2 inner product and L^2 norm are defined as

$$(\mathbf{u}, \mathbf{v}) = h^d \sum_{a \in \Lambda} u_a v_a \qquad \forall \mathbf{u}, \mathbf{v} \in V_\Lambda, \tag{1.5}$$

$$\|\mathbf{v}\| = (\mathbf{v}, \mathbf{v})^{1/2}, \ \forall \mathbf{v} \in V_\Lambda. \tag{1.6}$$

and discrete maximum norm is defined as

$$\|\mathbf{v}\|_\infty = \max_{a \in \Lambda} |v_a|, \qquad \forall \mathbf{v} \in V_\Lambda \tag{1.7}$$

The inverse of $\tanh(\cdot)$ is denoted by $\phi(\cdot)$ so that $\phi(r) = \frac{1}{2} \ln \frac{1+r}{1-r}$.

We introduce the homogeneous 'free energy' functions for $r \in (-1, 1)$

$$\psi(r) := \frac{1}{2}((1+r)\ln(1+r) + (1-r)\ln(1-r)) \tag{1.8}$$

Then as noted by Penrose[1], an important feature of the system (1.1) is the existence of a Lyapunov functional given in our notation by

$$I(\mathbf{u}) := \frac{\beta}{2}(\mathbf{A}\,\mathbf{u}, \mathbf{u}) + (\mathbf{e}, \psi(\mathbf{u})) \tag{1.9}$$