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PIECEWISE RATIONAL APPROXIMATIONS OF REAL ALGEBRAIC CURVES*

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Abstract

We use a combination of both algebraic and numerical techniques to construct a C^1-continuous, piecewise (m, n) rational ϵ-approximation of a real algebraic plane curve of degree d. At singular points we use the classical Weierstrass Preparation Theorem and Newton power series factorizations, based on the technique of Hensel lifting. These, together with modified rational Padé approximations, are used to efficiently construct locally approximate, rational parametric representations for all real branches of an algebraic plane curve. Besides singular points we obtain an adaptive selection of simple points about which the curve approximations yield a small number of pieces yet achieve C^1 continuity between pieces. The simpler cases of C^{-1} and C^0 continuity are also handled in a similar manner. The computation of singularity, the approximation error bounds and details of the implementation of these algorithms are also provided.

1. Introduction

An algebraic plane curve $\mathbf{C}$ of degree d in $\mathbb{R}^2$ is implicitly defined by a single polynomial equation $f(x, y) = 0$ of degree d with coefficients in $\mathbb{R}$. A rational algebraic curve of degree d in $\mathbb{R}^2$ can additionally be defined by rational parametric equations which are given as $(x = G_1(u), y = G_2(u))$, where G_1 and G_2 are rational functions in u of degree d, i.e., each is a quotient of polynomials in u of maximum degree d with coefficients in $\mathbb{R}$. Rational curves are only a subset of implicit algebraic curves of degree $d+1$. While all degree two curves (conics) are rational, only a subset of degree three (cubics) and higher degree curves are rational. In general, a necessary and sufficient condition for the global rationality of an algebraic curve of arbitrary degree is given by the Cayley-Riemann criterion: a curve is rational if and only if $g = 0$, where g, the

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genus of the curve is a measure of the deficiency of the curve's singularities from its maximum allowable limit [?, ?].

The Rational Approximation Problem

Given a real algebraic plane curve **C**: $f(x, y) = 0$ of degree d and of arbitrary genus, a box B defined by $\{(x, y)|\alpha \le x \le \beta, \gamma \le y \le \delta\}$, an error bound $\epsilon > 0$, and integers m, n with $m + n \le d$ construct a C^{-1}, C^0 or C^1 continuous piecewise rational ϵ-approximation of all portions of **C** within the given bounding box B, with each rational function $\frac{P_i}{Q_i}$ of degree $P_i \le m$ and degree $Q_i \le n$. Here C^{-1} means no continuity condition is imposed between the different pieces, C^0 implies there are no gaps and C^1 implies that the first derivatives are continuous at the common end points of adjacent pieces. The ϵ-approximation here means that the approximation error is within given ϵ. The input curve $f(x, y) = 0$ may be reducible and have several real components but we assume it has components of only single multiplicity, i.e. polynomial $f(x, y)$ has no repeated factors.

Results

We use a combination of both algebraic and numerical techniques to construct a C^1-continuous, piecewise (m, n) rational ϵ-approximation by two different approaches, of a real algebraic plane curve. At singular points we rely on the classical resolution of plane curves [?, ?] based on the Weierstrass Preparation Theorem [?] and Newton power series factorizations[?], using the technique of Hensel lifting[?]. These, together with modified Padé approximations, are used to efficiently construct locally approximate, rational parametric representations for all real branches of an algebraic plane curve. Besides singular points we obtain an adaptive selection of simple points about which the curve approximations yield a small number of pieces yet achieve C^1 continuity between pieces. The simpler cases of C^{-1} and C^0 continuity are also handled in a similar manner. The computation of singularity, the approximation error bounds and details of the implementation of these algorithms are also provided.

Applications

In geometric design and computer graphics one often uses rational algebraic curves and surfaces because of the advantages obtained from having both the implicit and rational parametric representations [?], [?]. While the rational parametric form of representing a curve allows efficient tracings, ease for transformations and shape control, the implicit form is preferred for testing whether a point is on the given curve, is on the left or right of the curve and is further conducive to the direct application of algebraic techniques. Simpler algorithms are also possible when both representations are available. For example, a straightforward method exists for computing curve - curve and surface - surface intersection approximations when one of the curves, respectively surfaces, is in its implicit form and the other in its parametric form. Global parameterization algorithms exist for implicit algebraic curves of genus zero [?, ?] which allows one to compute this dual representation. A solution to our rational approximation problem yields a rational representation, although approximate, and with all the above advantages for arbitrary genus algebraic plane curves. Perhaps even more important, there are requirements to approximate the algebraic curves in a computer aided geometric