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A MULTIGRID METHOD FOR NONLINEAR PARABOLIC PROBLEMS*1)

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Abstract

The multigrid algorithm in [13] is developed for solving nonlinear parabolic equations arising from the finite element discretization. The computational cost of the algorithm is approximate $O(N_kN)$ where N_k is the dimension of the finite element space and N is the number of time steps.

1. Introduction

The finite element methods for solving nonlinear parabolic problems are studied by many authors, such as Douglas and Dupont[5], Wheeler[4], Luskin[3], etc. They proposed various ways of computing the problems and proved the optimal order convergence rates of the methods, such as the linearized methods, the predictor-corrector methods, the extrapolation methods, the alternating direction methods and the iterative methods[2], etc. The multigrid methods for solving parabolic problems are studied by some authors, such as Hachbusch[14,15], Bank and Dupont[12], Brandt and Greenwald[16] as well as Yu[13]. But these methods are given mainly for linear parabolic equations. For nonlinear parabolic problems Hachbusch and Brandt in [14], [15], [16] gave the multigrid methods by using the integral differential equation and the frozen-τ technique.

In this paper we present a multigrid procedure for two-dimension nonlinear parabolic problems. The method is an extension of our earlier algorithm in [13] for linear parabolic problems. The iterative methods for solving the system of nonlinear algebraic equations are avoided because the unknown function $U_k^{n+\theta}$ in the nonlinear coefficient $a(x, U_k^{n+\theta})$ and the right term $f(x, t, U_k^{n+\theta})$ in the system of nonlinear algebraic equations is replaced by $I_kU_{k-1}^{n+\theta}$ in the multigrid procedure, where I_k denotes an intergrid transfer operator, θ a weighted function and $U_{k-1}^{n+\theta}$ the solutions of the equation in the (k-1)th level. We analyze the convergence of our algorithm and the computational cost of N

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time steps. The asymptotically computational cost is $O(NN_k)$ where N_k is the dimension of the discrete finite element space and N is the number of time steps. In addition, the methods can be applied to more general nonlinear parabolic problems.

The paper is organized as follows. In Section 2, we give the basic assumptions and properties by using of the finite element discretizing a nonlinear parabolic equation. In Section 3 we extend the time-dependent fully multigrid algorithm in [13] to the nonlinear parabolic equation. In Section 4 we analyze the convergence of the algorithm and in Section 5 we consider the computational cost and the development.

2. Notations and Preliminaries

We consider nonlinear parabolic initial value problems as follows:

$$\{ \partial u \partial t = \nabla(a(x,u)\nabla u)+f(x,t,u),\ (x,t) \in \Omega \times [0,T], u(x,t)=0,\ (x,t) \in \partial\Omega \times [0,T], u(x,0)=u_0(x),\ x \in \Omega, 2.1$$

where $\Omega \subset R^2$ is a convex polygonal domain, ∇ is a gradient operator on $x=(x_1,x_2)$ directions. Assume that the nonlinear coefficient $a(x,p)$ satisfies the condition: there are constants $K_0,\ K_1 > 0$ such that

$$0 < K_0 \le a(x,u) \le K_1,\ \forall (x,p) \in \bar{\Omega} \times R^1 .2.2$$

$a(x,p)$ and $f(x,t,p)$ hold uniformly Lipschitz condition with respect to p, i.e., there is a constant $L > 0$ such that

$$|a(x,p_1)-a(x,p_2)| \le L|p_1-p_2|,\ \forall (x,p) \in \bar{\Omega} \times R^1, |f(x,t,p_1)-f(x,t,p_2)| \le L|p_1-p_2|,\ \forall (x,t,p) \in \bar{\Omega} \times [0,T] \times R^1 .2.3$$

Further assume that for any $t \in [0,T],\ f(x,t,0) \in L^2(\Omega)$. Thus by (2.3), we have

$$|f(x,t,v(x,t))| \le |f(x,t,0)| + L|v(x,t)| \in L^2(\Omega),\ \forall v(x,t) \in L^2(\Omega).$$

The variational form of problem (2.1) is : Find a continuously differentiable mapping $u(t) = u(x,t) : [0,T] \to H_0^1(\Omega)$ such that

$$\{ (\ \partial u \partial t, v) + a(u;u,v) = (f(u),v), (u(x,0),v) = (u_0(x),v),\ \forall v \in H_0^1(\Omega) .2.4$$

where $a(u;u,v) = \int_\Omega a(x,u)\nabla u \nabla v dx,\ (f(u),v) = \int_\Omega f(x,t,u) v dx.$

Under the assumptions (2.3) and (2.4), a solution of the variational problem (2.4) such that $\|\nabla u\|_{L^\infty(L^\infty)} < +\infty$, if it exists, must be unique where $\|\nabla u\|_{L^\infty(L^\infty)}$ is defined by

$$\|\nabla u\|_{L^\infty(L^\infty)} = \|\|\nabla u\|_{L^\infty(\Omega)}\|_{L^\infty[0,T]}.$$

In the following we assume that a solution of the problem (2.4) exists and is unique. And the solution is smooth enough for the finite element analysis.

Let Γ be a mesh partition of the domain Ω (the triangulation or quadrilateral partition) which satisfies the partition quasi-uniformity conditions [17]. Since Ω is a