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THE STABILITY ANALYSIS OF THE θ-METHODS FOR DELAY DIFFERENTIAL EQUATIONS*

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Abstract

This paper deals with the stability analysis of $\theta-$methods for the numerical solution of delay differential equations (DDEs). We focus on the behaviour of such methods in the solution of the linear test equation $y'(t) = a(t)y(t) + b(t)y(t-\tau)$, where $\tau > 0$, $a(t)$ and $b(t)$ are functions from R to C. It is proved that the linear $\theta-$method and the one-leg $\theta-$method are TGP-stable if and only if $\theta = 1$.

1. Introduction

This paper deals with the numerical solution of the following initial- value problems

$$\begin{cases} y'(t) &= f(t, y(t), y(t-\tau(t))) \qquad t \geq 0, \\ y(t) &= \phi(t) \qquad t \leq 0, \end{cases} \tag{1.1}$$

where $y : R \to C$, $\tau(t) \geq 0$ is the delay term , $\phi(t) : R \to C$ is the initial function, whereas $y(t)$ is unknown for $t > 0$.

Let us consider the following linear delay differential equation:

$$\begin{cases} y'(t) &= ay(t) + by(t-\tau) \qquad t \geq 0 \\ y(t) &= \varphi(t) \qquad -\tau \leq t \leq 0, \end{cases} \tag{1.2}$$

where $y : R \to C$, a, b are complex, $\tau > 0$ is a constant delay. $\varphi(t)$ denotes a given function on $[-\tau, 0]$.

It is well-known that (see[1,2]), if $\varphi(t)$ is continuous and if

$$|b| < -\mathrm{Re}(a), \tag{1.3}$$

then the solution $y(t)$ to (1.2) tends to zero as $t \to \infty$ for every $\tau > 0$. In this case the solution $y(t)$ to (1.2) is called asymptotically stable.

Concerning numerical solution of (1.2), let's recall Barwell's (see[3]) definitions of P- and GP-stability .

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Definition 1.1. *A numerical method for DDEs is called P-stable if, for all coefficients* a, b *satisfying* (1.3)*, the numerical solution* $y_n \sim y(t_n)$ *of* (1.2) *at the mesh points* $t_n = nh$*,* $n \geq 0$*, satisfies*

$$y_n \to 0, \quad as \quad n \to \infty,$$

for every stepsize h *such that* $h = \tau/m$*,* m *is a positive integer.*

Definition 1.2. *A numerical method for DDEs is called GP-stable if, under condition* (1.3)*,* $y_n \to 0$ *, as* $n \to \infty$ *for every stepsize* $h > 0$*.*

Consider the following linear test equation which was introduced in [9]:

$$\begin{cases} y'(t) & = a(t)y(t) + b(t)y(t-\tau) \qquad t \geq 0, \\ y(t) & = \phi(t) \qquad -\tau \leq t \leq 0, \end{cases} \tag{1.4}$$

where $y : [-\tau, +\infty) \to C$, $a, b : [0, +\infty) \to C$ and $\tau > 0$, and the solution $y(t)$ of (1.4) is bounded by $max_{-\tau \leq t \leq 0} |\phi(t)|$, provided that, for every $t \geq 0$,

$$|b(t)| \leq -\mathrm{Re}(a(t)). \tag{1.5}$$

In [9], Torelli introduced two definitions of stability based on the test equation (1.4) as follows:

Definition 1.3. *A numerical method for DDEs is said to be PN-stable if, under the condition* (1.5)*, the numerical solution* y_n *of* (1.4) *is such that*

$$|y_n| \leq \max_{-\tau \leq t \leq 0} |\phi(t)| \tag{1.6}$$

for every $n \geq 0$ *and for every stepsize* $h = \tau/m$*, where* m *is a positive integer.*

Definition 1.4. *A numerical method for DDEs is called GPN-stable if, under the condition* (1.5)*, the numerical solution of* (1.4) *satisfies* (1.6) *for every* $n \geq 0$ *and for every stepsize* $h > 0$*.*

The numerical stability of θ-methods and Runge-Kutta methods have been widely investigated in [5,7,8,12]. The numerical stability of the $\theta-$ methods with respect to the linear test equation (1.2) have been carefully studied in [7]. In [9], Torelli has dealt with numerical stability based on Definition 1.3 and 1.4 of the $\theta-$methods with respect to the linear test equation (1.4).

It is the purpose of this paper to investigate the asymptotic stability behaviour of the theoretical solution and the numerical solution of (1.4). In section 2 , we derive a sufficient condition for (1.4) such that the solution to (1.4) is asymptotically stable. In section 3 and section 4, it is proven that the linear $\theta-$method and the one-leg $\theta-$method are TGP-stable if and only if $\theta = 1$.

2. Asymptotic Stability of the Theoretic Solution of DDEs

First of all, let us consider the following nonlinear systems

$$\begin{cases} y'(t) & = f(t, y(t), y(t-\tau)) \qquad t \geq 0, \\ y(t) & = \phi(t) \qquad t \leq 0, \end{cases} \tag{2.1}$$