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MODIFIED DISCREPANCY PRINCIPLES WITH PERTURBED OPERATORS AND NOISY DATA*1)

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Abstract

We investigate a class of a posterior parameter choice for iterated Tikhonov regularization with perturbed operators and noisy data by using modified Arcangeli's method. The rate of convergence of regularization approximation is achieved.

1. Introduction

Let X, Y be real Hilbert spaces, $T : X \to Y$ a bounded linear operator with nonclosed range $R(T), y \in D(T^+) = R(T)\dot{+}R(T)^\perp$, where T^+ is the Moore-Penrose inverse of T[1]. For each $\delta > 0$, let $y_\delta \in Y$ be such that

$$\|y - y_\delta\| \leq \delta. \tag{1}$$

As we know, the problem of solving the operator equation of the first kind

$$Tx = y \tag{2}$$

is, in generality, ill-posed[2]. Also we can not ensure that T^+y_δ is a reasonable approximation of T^+y since T^+ is a unbounded operator. In practice, one tries to construct a stable approximate solution to the equation (2) by regularization methods. A well-known regularization method for approximating T^+y is the Tikhonov regularization method[3]. For each $\delta > 0$ and $\alpha > 0$, we denote by $x_{\alpha,\delta}$ the Tikhonov regularization approximation for T^+y. A crucial problem is the choice of regularization parameter α in dependence of the noisy level δ leading to optimal convergence rates.

The earliest methods of this type are Morozov's discrepancy principles, where α is chosen such that

$$\|Tx_{\alpha,\delta} - y_\delta\|^2 = \delta^2,$$

and Arcangeli's method, where α is chosen as the root of[3]

$$\|Tx_{\alpha,\delta} - y_\delta\|^2 = \frac{\delta^2}{\alpha}\ .$$

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However, it is shown that neither Morozov nor Arcangeli's method yields the optimal convergence rates[4,5]. In order to improve the convergence rates, J.T. King probes the iterated Tikhonov regularization[6]. In [7], H.W. Engl proposes the Modified Arcangeli's method of choosing the regularization parameter that leads to higher rates of convergence.

All the methods and results above-mentioned are only applicable if T is exactly known. However, in practice, not only the right-hand member of equations but operators is approximately given. Z.Y. Hou and H.N. Li have shown two ways of choosing regularization parameter for Tikhonov regularization with perturbed operators and noisy data, and obtained the results concerning the rates of convergence[8,9]. For more information in this area, we refer the reader to [10,11].

The aim of this paper is to provide a posteriori parameter choice and higher asymptotic convergence rate for iterated Tikhonov regularization with not only noisy data but perturbed operators by using modified Arcangeli's method. The result described in the paper has much more practical value.

2. Modified Arcangeli's Principles

For real number $h > 0$, let $T_h : X \to Y$ be a bounded linear operator such that

$$\|T - T_h\| \le h. \tag{3}$$

To exclude trivialitive, from now on, we assume that

$$y \in R(T), \qquad T^* y \neq 0, \qquad T^* y_\delta \neq 0, \tag{4}$$

where T^* is the adjoint of T. For all $j \in N$, let $x^{(j)}_{\alpha,\delta,h}$ be the result of iterated Tikhonov regularization of order j , i.e,

$$x^{(0)}_{\alpha,\delta,h} = 0 \ , \qquad x^{(j)}_{\alpha,\delta,h} = (T_h^* T_h + \alpha I)^{-1}(T_h^* y_\delta + \alpha x^{(j-1)}_{\alpha,\delta,h}) \ , \tag{5}$$

where N denotes the set of natural number, α is a positive and real number and I is the identity operator. It follows by induction that for any $j \in N$,

$$x^{(j)}_{\alpha,\delta,h} = \sum_{i=1}^{j} \alpha^{i-1} (T_h^* T_h + \alpha I)^{-i} T_h^* y_\delta. \tag{6}$$

For simplicity of notation we replace $T_h^* T_h, T_h T_h^*, T^* T$ and TT^* by $\tilde{T}_h, \hat{T}_h, \tilde{T}$ and $\hat{T}$, respectively, below. It is easy to see that the equality

$$T^* g(\hat{T}) = g(\tilde{T}) T^*$$

holds if the function $g(t)$ is continuous on $[0, +\infty)$.

Lemma 1. *Let*

$$\rho_j(\alpha) = \left\| \tilde{T}_h x^{(j)}_{\alpha,\delta,h} - T_h^* y_\delta \right\|^2 \tag{7}$$