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A SMALLEST SINGULAR VALUE METHOD FOR SOLVING INVERSE EIGENVALUE PROBLEMS *1)

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Abstract

Utilizing the properties of the smallest singular value of a matrix, we propose a new, efficient and reliable algorithm for solving nonsymmetric matrix inverse eigenvalue problems, and compare it with a known method. We also present numerical experiments which illustrate our results.

1. Introduction

Consider the following inverse eigenvalue problem:

Problem G. *Let $A(x) \in \mathbf{R}^{n\times n}$ be a real analytic matrix-valued function of $x \in \mathbf{R}^n$. Find a point $x^* \in \mathbf{R}^n$ such that the matrix $A(x^*)$ has a given spectral set $L = \{\lambda_1, \cdots, \lambda_n\}$. Here $\lambda_1, \cdots, \lambda_n$ are given complex numbers and closed under complex conjugation.*

This kind of problem arises often in various areas of applications (see Freidland et al.(1987) and references contained therein). The two special cases of Problem G, which are frequently encountered, are the following problems proposed by Downing and Householder(1956):

Problem A. *Let A be a given $n \times n$ real symmetric matrix, and $\lambda_1, \cdots \lambda_n$ be n given real numbers. Find an $x^* = (x_1^*, \cdots, x_n^*)^T \in \mathbf{R}^n$ such that the matrix $A + D(x^*)$ has eigenvalue $\lambda_1, \cdots \lambda_n$. Here $D(x^*) = diag(x_1^*, \cdots, x_n^*)$.*

Problem M. *Let A be a given $n \times n$ real symmetric positive definite matrix, and $\lambda_1, \cdots \lambda_n$ be n given positive numbers. Find an $x^* = (x_1^*, \cdots, x_n^*)^T \in \mathbf{R}^n$ with $x_i^* > 0$ such that the matrix $AD(x^*)$ has eigenvalue $\lambda_1, \cdots \lambda_n$.*

Problem A and M are known as the additive inverse eigenvalue problem and the multiplicative inverse eigenvalue problem, respectively.

There are large literature on conditions for existence and uniqueness of solutions to Problem G in many special cases (see Xu(1989) and references contained therein). Here, we will assume that Problem G has a solution and concentrate on how to compute it numerically.

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Although many numerical methods for solving various special cases of Problem G have been proposed, only one, which is proposed by Biegler-König(1981), can be applied to the nonsymmetric case of $A(x)$. In this paper we propose a new, efficient and reliable algorithm for solving Problem G, which is based on the properties of the smallest singular value of a matrix, therefore, called a smallest singular value method. This algorithm also has no restriction of symmetry, and is more efficient and reliable than Biegler-König's algorithm.

Notation. Throughout this paper we use the following notation. $\mathbf{R}^{m\times n}$ is the set of all $m \times n$ real matrices, and $\mathbf{C}^{m\times n}$ the set of all $m \times n$ complex matrices. $\mathbf{R}^n = \mathbf{R}^{n\times 1}$ and $\mathbf{C}^n = \mathbf{C}^{n\times 1}$. I is the $n \times n$ identity matrix. The superscript T and H are for transpose and conjugate transpose, respectively. $\det(A)$ and $\mathrm{tr}(A)$ denote the determinant and the trace of a matrix A, respectively. $\sigma_{\min}(A)$ denotes the smallest singular value of a matrix A. The norm $\|x\|$ stands for the usual Euclidean norm of vector x, and $\|x\|_\infty$ for the max-norm of vector x.

2. Formulation of the Numerical Methods

We will now describe two methods for solving Problem G in the case where the given eigenvalues are distinct. Assume there exists a solution x^* to Problem G. In some neighborhood of x^* we will first consider the following formulation of Problem G:

Formulation I. Solve the nonlinear system

$$f(x) = (f_1(x), \cdots, f_n(x))^T = 0, \tag{2.1}$$

where

$$f_i(x) = \sigma_{\min}(A(x) - \lambda_i I) \tag{2.2}$$

for $i = 1, 2, \cdots, n$.

In order to apply Newton's method to (2.1), we need the partial derivatives of $f(x)$ with respect to the $x_1, \cdots, x_n$. To calculate these derivatives we apply Sun's Theorem [9]:

Theorem. *Let $x \in \mathbf{R}^l$ and $B(x) \in \mathbf{C}^{m\times n}$. Suppose that $Re(B(x))$ and $Im(B(x))$ are real analytic matrix-valued function of x. If σ is a simple non-zero singular value of $B(x^{(0)})$, $v \in \mathbf{C}^n$ and $u \in \mathbf{C}^m$ are associated unit right and left singular vectors, respectively, then there exists a simple singular value $\sigma(x)$ of $B(x)$ which is a real analytic function of x in some neighborhood of $x^{(0)}$, and*

$$\sigma(x^{(0)}) = \sigma, \qquad \frac{\partial \sigma(x^{(0)})}{\partial x_j} = Re\Big(u^H \frac{\partial B(x^{(0)})}{\partial x_j} v\Big).$$

Utilizing the above theorem, if $f_i(x) \neq 0$, we get

$$\frac{\partial f_i(x)}{\partial x_j} = \mathrm{Re}\Big(u_i(x)^H \frac{\partial A(x)}{\partial x_j} v_i(x)\Big), \tag{2.3}$$