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MATRIX ANALYSIS TO ADDITIVE SCHWARZ METHODS*[1)]

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Abstract

Matrix analysis on additive Schwarz methods as preconditioners is given in this paper. Both cases of with and without coarse mesh are considered. It is pointed out that an advantage of matrix analysis is to obtain more exact upper bound. Our numerical tests access the estimations.

1. Introduction

We consider the following second order elliptic boundary value problem:

$$\mathcal{L}u = f, \quad \text{in} \quad \Omega, \tag{1}$$

$$u = 0, \quad \text{on} \quad \partial\Omega, \tag{2}$$

where $\mathcal{L}$ is a self-adjoint positive operator and

$$\Omega \subset \mathcal{R}^d \quad (1 \le d \le 3)$$

is a polyhedral domain.

A weak solution has the following form: Find u $\in H_1^0(\Omega)$ such that :

$$\mathcal{A}(u, v) = f(v), \quad \forall v \in H_1^0(\Omega)$$

$$\mathcal{A}(u, v) = \int_\Omega \mathcal{L}u(x)v(x)dx, \quad f(v) = \int_\Omega f(x)v(x)dx.$$

Let $V^h := \mathcal{M} = \text{Span } \{\phi_i\}$, where $\{\phi_i\}$ could be nodal basis consisting of piece-wise linear functions or other spline functions. Substituting the following solution

$$u^h = \sum u_i \phi_i$$

into the above weak form leads to a discrete equation

$$Au = f, \tag{3}$$

where

$$A = (\alpha_{ij}), \quad \alpha_{ij} = \mathcal{A}(\phi_i, \phi_j). \tag{4}$$

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It is well known that the coefficient matrix A is symmetry positive definite matrix with condition number
$$\kappa(A) = O(h^{-2}). \tag{5}$$
When we use conjugate gradient algorithm for solving the system, for a given tolerance, the iteration number will proportional to h^{-1}. This convergent rate is really slow for a large scale problems. It is our purpose, in this paper, to do some analysis on Additive Schwarz Methods (ASM) as preconditioners in detail. In order to obtain estimation on condition number of the preconditioner system more accuracy, we take 1-D case as a model problem. The related results on higher dimension will be reported later.

2. A Projector Preconditioner

Suppose a subspace
$$\mathcal{M}_c := \mathrm{Span}\{\psi_k\} \subset \mathcal{M}$$
with the basis transformation
$$\psi_k = \sum t_{ki}\phi_i, \quad \Psi = T\Phi, T = (t_{ki}).$$
Define a projector $P_c : \mathcal{M} \to \mathcal{M}_c$ such that for any given $u \in \mathcal{M}$
$$A(P_c u, v) = A(u, v), \quad \forall v \in \mathcal{M}_c. \tag{6}$$
Assume that
$$P_c\phi_j = \sum \beta_{kj}\psi_k \ \text{ or } \quad P_c\Phi = G\Psi, \quad G = (\beta_{kj}).$$
So
$$A(P_c\phi_j, \psi_l) = \sum A(\psi_k, \psi_l)\beta_{kj} \ .$$
Denote
$$A_c = (A(\psi_k, \psi_l)), \qquad Q = (A(\phi_j, \psi_l)),$$
then
$$A_c G = Q.$$
This means that as a linear operator from $\mathcal{M}$ to $\mathcal{M}_c$, the matrix representation of P_c from coordinate basis ϕ to ψ is as follows
$$P_c \sim \quad G = A_c^{-1}Q = A_c^{-1}TA.$$
When we back to the original space and take P_c as a linear operator from $\mathcal{M}$ to $\mathcal{M}$ itself, the corresponding matrix form becomes
$$P_c \sim \quad T'G = A_c^{-1}Q = T'A_c^{-1}TA \ . \tag{7}$$
Therefore, we may look the projector P_c as the result from a preconditioned operator of A, the related preconditioner is
$$B_c := T'A_c^{-1}T \ . \tag{8}$$