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OPTIMAL-ORDER PARAMETER IDENTIFICATION IN SOLVING NONLINEAR SYSTEMS IN A BANACH SPACE*)

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Abstract

We study the sufficient and necessary conditions of the convergence for parameter-based rational methods in a Banach space. We derive a closed form of error bounds in terms of a real parameter λ $(1 \le \lambda < 2)$. We also discuss some behaviors when the family is applied to abstract quadratic functions on a Banach space for $\lambda = 2$.

1. Introduction

We consider the problem of solving

$$F(x) = 0, \tag{1}$$

where $F : D \subset X \to Y$ is a nonlinear differential operator defined on some convex subset D of a Banach space X with values in a Banach space Y. Many problems of applied mathematics can be brought in the form of equation (1). (see Ortega and Rheinboldt [1970], Lancaster [1977], Dennis and Schnabel [1983], Cuyt and Rall [1985], Laub [1991], etc.) A well-known method for solving (1) is the third-order Halley. Given an approximation x_k, compute x_{k+1} by

$$x_{k+1} = x_k - [F'(x_k) - \frac{1}{2}F''(x_k)F'(x_k)^{-1}F(x_k)]^{-1}F(x_k), \tag{2}$$

Recent years, Kantorovich-type convergence (sufficient conditions for the convergence) of the Halley method in Banach space setting has been mentioned by many authors: Candela and Marquina [1990], and Kanno [1992]. In this paper, we introduce a real parameter λ and design a new parameter-based rational iterations in Banach spaces as follows:

$$y_k = x_k - F'(x_k)^{-1}F(x_k)$$

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$$H(x_k, y_k) = F'(x_k)^{-1}F''(x_k)(y_k - x_k)$$

$$x_{k+1} = y_k - \frac{1}{2}H(x_k, y_k)[I + \frac{\lambda}{2}H(x_k, y_k)]^{-1}(y_k - x_k), \tag{3}$$

which include the Halley method as a specific choice of the parameter. We will not only provide a complete Kantorovich-type convergence analysis as well as a local convergence for this one-parameter family for $1 \le \lambda < 2$ but also we point out that the maximum order of convergence for the iteration at $\lambda = 2$ is greater than the famous conjecture by Traub [15]. The conjecture states that their maximum order of convergence is three, but we will show that it is of order four.

2. Sufficient Conditions for the Convergence

We first need a lemma.

Lemma 2.1. *Let $F(x)$ be a nonlinear operator from an open convex domain D in a Banach space X to another Banach space Y. Suppose that F has 2nd order continuous Frechet derivatives on D. Then the $F(x_{k+1})$ together with the sequence $\{x_k\}_{k=0}^{\infty}$ generated by (3) has the following approximation for all $k \ge 0$ and $1 \le \lambda \le 2$,*

$$F(x_{k+1}) = \int_0^1 F''[y_k + t(x_{k+1} - y_k)](1-t)dt(x_{k+1} - y_k)^2 - \frac{1}{2}\int_0^1 [F''[x_k + t(y_k - x_k)]$$
$$[1 - \lambda(1-t)]dt(y_k - x_k)H(x_k, y_k)[I + \frac{\lambda}{2}H(x_k, y_k)]^{-1}(y_k - x_k)$$
$$+ \int_0^1 \{F''[x_k + t(y_k - x_k)](1-t) - \frac{1}{2}F''(x_k)\}dt(y_k - x_k)$$
$$\times [I + \frac{\lambda}{2}H(x_k, y_k)]^{-1}(y_k - x_k). \tag{4}$$

Now we can state our main result.

Theorem 2.1. *Let $F(x) : D \subset X \to Y$, X and Y are real or complex Banach spaces, and D is an open convex domain. Assume that F has 2nd order continuous Frechet derivatives on D and satisfies the following standard Newton-Kantorovich conditions:*

$$\| F''(x) \| \le M, \| F''(x) - F''(y) \| \le N \| x - y \|, \ for\ all\ x, y \in D. \tag{5}$$

For a given initial value $x_0 \in D$, assume that $F'(x_0)^{-1}$ exists and satisfies

$$\| F'(x_0)^{-1} \| \le \beta, \ \ \| F'(x_0)^{-1}F(x_0) \| \le \eta, \tag{6}$$

$$M[1 + \frac{2N}{3(2-\lambda)M^2\beta}]^{1/3} \le K, \ 1 \le \lambda < 2, \tag{7}$$

$$h = K\beta\eta \le 0.5, \tag{8}$$

$$\overline{S(x_0, t^*)} \subset D, \tag{9}$$