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MODIFIED UPWIND TAYLOR FINITE ELEMENT SCHEMES FOR 1-D CONSERVATION LAWS I. A BASIC IDEA*1)

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Abstract

In this paper, first, modified upwind finite element schemes are presented for two-point value problem, and then a class of modified upwind Taylor finite element schemes are derived for one dimensional linear hyperbolic equation. The main point of the paper is how to consider the upwind property to construct base functions to make the schemes obtained be MmB (or TVD). Numerical experiments are given to show that the method is efficient to solve the discontinuous solutions.

1. Introduction

Many numerical methods mainly contribute to solve nonlinear hyperbolic conservation laws and efficiently make the methods to suit for solving discontinuous solutions, that is, numerical solutions have high resolution, high order accurate and non-oscillatory properties. Let us recall some methods to treat these things: First of all, we can say that the development of finite difference method is divided into two steps, the first step is called classical difference methods, such as, Lax-Friedrichs scheme, Godunov scheme, Lax-Wendroff scheme and so on; the other is called modern methods, for example, some TVD type schemes[1,2], MUSCL schemes[3], ENO (or UNO) schemes[4,5], PPM scheme[6], MmB schemes[7] etc.; The second of the methods should be finite element method and spectral method, and we can say that both finite element and spectral methods are far beyond finite difference methods to treat discontinuities for nonlinear hyperbolic equations, although some schemes, such as, characteristic Galerkin method [8] and modified characteristic Petrov-Galerkin method [9], discontinuity finite element method [10], finite element method based on stream lines [11] and so on, have been presented, and to solve discontinuous solutions, these methods are modified or combined with modern techniques from the work of finite difference methods.

In order to develop finite element method to be suit for solving nonlinear hyperbolic equations both widely and efficiently, in this paper a class of modified upwind Taylor

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finite element schemes are constructed for one dimensional linear hyperbolic equation. The methods is mainly to consider the constructions of base functions: firstly, form a base function, which depends on the characteristic (or upwind) property of the model; secondly, modify the base function obtained and give a nonlinear base function. Hence a nonlinear element is proposed as in [9]. Then we use Taylor expansion in time direction and finite element method in space direction. So in this paper, modified upwind finite element schemes are presented for two-point boundary problem in section 2, and in section 3, a class of modified upwind Taylor finite element schemes are derived for linear hyperbolic equation; Finally, some numerical experiments are given for Riemann initial value problems.

2. Modified Upwind FES for Two-Point Boundary Value Problem

Consider the following boundary value problem,

$$K\frac{d^2u}{dx^2} - V\frac{du}{dx} = 0, \quad x \in (0,1) \tag{2.1}$$

$$u(0) = u_0, \qquad u(1) = u_1. \tag{2.2}$$

When K/V is sufficient small, problem (2.1) (2.2) belongs to a singular perturbation problem and the solution of the problem produces boundary layer near point 0 or 1. The numerical solutions are required to have high resolution and non-oscillatory properties in the boundary layer regions, and have higher order approximate accuracy in smooth regions.

Let us see the weak form of (2.1) – (2.2),

$$K\int \frac{du}{dx}\frac{d\varphi}{dx} - V\int u\frac{d\varphi}{dx} = 0, \quad \forall \varphi \in C_0(I\!R) \tag{2.3}$$

Here we choose the following unit linear function as a base function

$$\varphi_1(x) = \begin{cases} 1 + \dfrac{x - x_i}{\Delta x}, & x \in (x_{i-1}, x_i) \\ 1 - \dfrac{x - x_i}{\Delta x} & x \in (x_i, x_{i+1}) \\ 0 & else \end{cases} \tag{2.4}$$

and set

$$u_h(x) = \sum_j u_j \varphi_j(x),$$

then we place $u_h(x)$ and φ_j in (2.3) and get

$$-\frac{K}{\Delta x}(u_{j+1} - 2u_j + u_{j-1}) - \frac{V}{2\Delta x}(u_{j+1} - u_{j-1}) = 0\ . \tag{2.5}$$

Scheme (2.5) is second order accurate, the solution will produce oscillatory phenomena near boundary layer when K/V is sufficient small.

In order to eliminate the oscillations, in [12] a modified base function is given by weighted function as follows,

$$\overline{\varphi}_j = \varphi_j + \alpha F_j(x)$$