Journal of Computational Mathematics, Vol.13, No.3, 1995, 232–242.

# HIGH-ACCURACY P-STABLE METHODS WITH MINIMAL PHASE-LAG FOR $y'' = f(t, y)$ *

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**Abstract**

In this paper, we develop a one-parameter family of P-stable sixth-order and eighth-order two-step methods with minimal phase-lag errors for numerical integration of second order periodic initial value problems:

$$y'' = f(t, y), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'.$$

We determine the parameters so that the phase-lag (frequency distortion) of these methods are minimal. The resulting methods are P-stable methods with minimal phase-lag errors. The superiority of our present P-stable methods over the P-stable methods in [1–4] is given by comparative studying of the phase-lag errors and illustrated with numerical examples.

## 1. Introduction

The development of numerical integration formulae for the direct integration of the periodic initial-value problem

$$y'' = f(t, y), \quad y(t_0) = y_0, \quad y'(t_0) = y_0' , \tag{1.1}$$

which arises in the theory of orbital mechanics and in the study of wave equations, has created considerable interest in the recent years.

Usually, the Numerov's method

$$y_{n+1} = 2y_n - y_{n-1} + \frac{h^2}{12}(f_{n+1} + 10f_n + f_{n-1}) \tag{1.2}$$

is the most popular method. Although, Numerov's method has phase-lag of order four and possess only a finite interval of periodicity $(0, 2.449^2)$. Recently Chawla and Rao[2,3] developed fourth-order and sixth-order P-stable methods with phase-lag of order six.

Ananthakrishnaiah[4] obtained a two-parameter family of second order P-stable methods $M_2(\alpha, \beta)$ with phase-lag of order six. It is therefore natural to ask whether we can obtain P-stable methods with phase-lag order and accuracy order higher than the

* Received November 25, 1993.

methods in [1–4]. The purpose of this paper is by modificating the methods in [1–4] and selecting parameters suitably, to obtain a new family of methods with sixth-order and eighth-order. Comparing with the methods in [1–4], our methods are more useful when a large step-size is used, that is , when a modest accuracy is sufficient or the solution which consists of a slowly varying oscillation with a high-frequency oscillation superimposed, has a small amplitude. At the end of this paper we give two examples to demonstrate that our methods are better than the methods in [1–4].

## 2. Basic Theory

When we apply an symmetry implicit two-step method to the test equation

$$y'' = -\lambda^2 y, \quad \lambda > 0 \ , \tag{2.1}$$

we obtain the polynomial

$$\Omega(\xi, H^2) = A(H)\xi^2 - 2B(H)\xi + A(H), \quad H = \lambda h \ . \tag{2.2}$$

It is stability and $\Omega(\xi, H^2) = 0$ is characteristic equation, $A(H)$ and $B(H)$ are polynomials of $H = \lambda h$.

**Definition 1.** (Lambert and Watson[5]) *The method with stability polynomial* (2.2) *is said to have interval of periodicity* $(0, H_p^2)$ *if for all* $H^2 \in (0, H_p^2)$*, the roots* $\xi_{1,2}$ *of* $\Omega(\xi, H^2)$ *satisfy*

$$\xi_{1,2} = e^{\pm i\theta(H)} \tag{2.3}$$

*for some real valued function* $\theta(H)$.

**Definition 2.** *The method with stability polynomial* (2.2) *is said to be P-stable if its interval of periodicity is* $(0, \infty)$.

It is easy to see that the roots of (2.2) are complex and of module one if

$$\Big|\frac{B(H)}{A(H)}\Big| < 1 \ . \tag{2.4}$$

Thus, the P-stability condition is satisfied if

$$A(H) + B(H) > 0 \text{ and } A(H) - B(H) > 0, \text{ for all } H^2 \in (0, \infty) \ . \tag{2.5}$$

The exact solution of the test equation (2.1) with the initial condition $y(t_0) = y_0$ and $y'(t_0) = y'_0$ is given by

$$y(t) = y_0 \cos \lambda t + \frac{y'_0}{\lambda} \sin \lambda t \ . \tag{2.6}$$

Evaluating (2.6) at $t_{n+1}, t_n$ and $t_{n-1}$ and eliminating $y_0$ and $y'_0$, we obtain

$$y(t_{n+1}) - 2\cos \lambda h y(t_n) + y(t_{n-1}) = 0 \ , \tag{2.7}$$