Journal of Computational Mathematics, Vol.20, No.5, 2002, 525–532.

ORDER RESULTS OF GENERAL LINEAR METHODS FOR MULTIPLY STIFF SINGULAR PERTURBATION PROBLEMS*1)

Si-qing Gan
(*Department of Computer Science and Technology, Tsinghua University, Beijing 100084, China*)
Geng Sun
(*Institute of Mathematics, Academia Sinica, Beijing 100080, China*)

Abstract

In this paper we analyze the error behavior of general linear methods applied to some classes of one-parameter multiply stiff singularly perturbed problems. We obtain the global error estimate of algebraically and diagonally stable general linear methods. The main result of this paper can be viewed as an extension of that obtained by Xiao [13] for the case of Runge-Kutta methods.

Key words: Singular perturbation problem; Stiffness; General linear method; Global error estimate.

1. Introduction

Let $\langle , \rangle$ be the standard inner product on real Euclidean space and $\|\cdot\|$ the corresponding norm. The matrix norm is subordinate to the vector norm $\|\cdot\|$. Consider the following singular perturbation problems(SPPs)

$$\begin{cases} x'(t) = f(x(t), y(t)), & t \in [0, T], \\ \epsilon y'(t) = g(x(t), y(t)), & 0 < \epsilon \ll 1, \end{cases} \tag{1.1}$$

with initial values $(x(0), y(0)) \in \check{G}$ admitting a smooth solution $(x(t), y(t))$ (i.e. all derivatives of $x(t)$ and $y(t)$ up to a sufficiently high order are bounded independently of the stiffness of the problem), where $\check{G}$ is an appropriate region on $R^M \times R^N$, and the mappings $f : \check{G} \to R^M$ and $g : \check{G} \to R^N$ are sufficiently smooth and satisfy the following conditions

$$\langle f(x_1, y) - f(x_2, y), x_1 - x_2 \rangle \leq \omega \|x_1 - x_2\|^2, \tag{1.2a}$$

$$\langle g(x, y_1) - g(x, y_2), y_1 - y_2 \rangle \leq -\|y_1 - y_2\|^2, \tag{1.2b}$$

$$\|f(x, y_1) - f(x, y_2)\| \leq L_1 \|y_1 - y_2\|, \tag{1.2c}$$

$$\|g(x_1, y) - g(x_2, y)\| \leq L_2 \|x_1 - x_2\|, \tag{1.2d}$$

with moderately-sized constants ω, L_1 and L_2.

We note that the one-sized Lipschitz condition (1.2a) is weaker than the conventional Lipschitz condition

$$\|f(x_1, y) - f(x_2, y)\| \leq L \|x_1 - x_2\|, \tag{1.3}$$

since (1.3) implies (1.2a) with $\omega = L$ for moderately-sized L. If the problem (1.1) satisfies (1.3) with moderately-sized L, then it is a singly stiff singular perturbation problem(SSPP) because its stiffness is only caused by the small parameter ϵ. For the problem (1.1) with $L \gg 1$, it is

* Received May 19, 2000.
1)This work is supported by the National Natural Science Fundation of China. (No. 19871086 & 10101027)

a multiply stiff singular perturbation problem(MSPP) whose stiffness is caused by the small parameter ϵ and some other factors.

Although stiff SPPs is considered as a special class of stiff initial value problems of ordinary differential equations, B-theory (cf. [3, 6, 9]) can't cover the former because of its very special structure. Recently, some developments of quantitative convergence analysis for numerical methods applied to SSPPs have been made (cf. [4, 5, 6, 11]). The main technique approach is a transformation of the system (1.1) into a series of semi-explicit differential-algebraic equtions by $\epsilon-$asymptotic expansions, in the meantime, the numerical solutions are also expanded analogously. In 1991, Lubich [10] obtained some quantitative convergence results of $A(\alpha)-$stable linear multistep methods for SSPPs by the other approach(i.e. direct approach). In 1999, Xiao [13] discussed convergence of one-leg methods and Runge-Kutta methods for MSPPs by direct approach. This paper is concerned with the error analysis of general linear methods for MSPPs by direct approach. We obtain the global error estimate of algebraically stable and diagonally stable general linear methods. Our main result (Theorem 3.3) can be considered as an extension of that obtained by Xiao [13].

2. General linear methods for SPPs

A $r-$step and $s-$stage general linear method(cf.[3, 9]) applied to (1.1) reads

$$X_i^{(n)} = h\sum_{j=1}^{s} C_{ij}^{11} f(X_j^{(n)}, Y_j^{(n)}) + \sum_{j=1}^{r} C_{ij}^{12} x_j^{(n-1)}, \quad i = 1, \ldots, s, \tag{2.1a}$$

$$\epsilon Y_i^{(n)} = h\sum_{j=1}^{s} C_{ij}^{11} g(X_j^{(n)}, Y_j^{(n)}) + \epsilon\sum_{j=1}^{r} C_{ij}^{12} y_j^{(n-1)}, \quad i = 1, \ldots, s, \tag{2.1b}$$

$$x_i^{(n)} = h\sum_{j=1}^{s} C_{ij}^{21} f(X_j^{(n)}, Y_j^{(n)}) + \sum_{j=1}^{r} C_{ij}^{22} x_j^{(n-1)}, \quad i = 1, \ldots, r, \tag{2.1c}$$

$$\epsilon y_i^{(n)} = h\sum_{j=1}^{s} C_{ij}^{21} g(X_j^{(n)}, Y_j^{(n)}) + \epsilon\sum_{j=1}^{r} C_{ij}^{22} y_j^{(n-1)}, \quad i = 1, \ldots, r, \tag{2.1d}$$

$$\xi_n^{(x)} = \sum_{j=1}^{r} \beta_j x_j^{(n)}, \qquad \xi_n^{(y)} = \sum_{j=1}^{r} \beta_j y_j^{(n)}, \tag{2.1e}$$

here $h > 0$ is the fixed stepsize, the coefficients C_{ij}^{IJ} and β_j are real constants, $X_i^{(n)}, x_i^{(n)}$ and $\xi_n^{(x)}$ are approximations to $x(t_n+\mu_i h), H_i^{(x)}(t_n+\nu_i h)$ and $x(t_n+\eta h)$, respectively, $Y_i^{(n)}, y_i^{(n)}$ and $\xi_n^{(y)}$ are approximations to $y(t_n+\mu_i h), H_i^{(y)}(t_n+\nu_i h)$ and $y(t_n+\eta h)$, respectively. $H_i^{(x)}(t_n+\nu_i h)$ and $H_i^{(y)}(t_n+\nu_i h)$ denote a piece of information about the true solution $x(t)$ and $y(t)$ respectively, i.e.

$$H_i^{(x)}(t) = a_i x(t) + hb_i x'(t), \qquad H_i^{(y)}(t) = a_i y(t) + hb_i y'(t), \qquad i = 1, \ldots, r,$$

a_i, b_i, μ_i, ν_i and η are real constants.

For any matrix H, let $\bar{H} = H \otimes I_M, \tilde{H} = H \otimes I_N$, where $\otimes$ denotes the Kronecker product of two matrices, I_l denotes an $l \times l$ unit matrix. Let $C_{IJ} = [C_{ij}^{IJ}]$ and $\beta = [\beta_1, \ldots, \beta_r]$, the process (2.1) can be written in more compact form

$$X^{(n)} = h\bar{C}_{11} F(X^{(n)}, Y^{(n)}) + \bar{C}_{12} x^{(n-1)}, \tag{2.2a}$$

$$\epsilon Y^{(n)} = h\tilde{C}_{11} G(X^{(n)}, Y^{(n)}) + \epsilon\tilde{C}_{12} y^{(n-1)}, \tag{2.2b}$$

$$x^{(n)} = h\bar{C}_{21} F(X^{(n)}, Y^{(n)}) + \bar{C}_{22} x^{(n-1)}, \tag{2.2c}$$