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DISCONTINUOUS FINITE ELEMENT METHOD FOR CONVECTION-DIFFUSION EQUATIONS*1)

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Abstract

A discontinuous finite element method for convection-diffusion equations is proposed and analyzed. This scheme is designed to produce an approximate solution which is completely discontinuous. Optimal order of convergence is obtained for model problem. This is the same convergence rate known for the classical methods.

Key words: Discontinuous finite element method, Convection-diffusion equations.

1. Introduction

The finite element approximation of the convection-diffusin equations has been investigated using several different approaches (see e.g. [3] [4] and the references therein). Previous analysis in primal formulation of these problems was done for two types of approximation schemes : one which produces a continuous piecewise polynomial approximation and one which produces a piecewise polynomial approximation which are continuous for certain number of moments accross interelement edges [2] (nonconforming approximation). All these finite element methods have optimal order of convergence, assuming sufficient regularity.

In this paper, we propose and analyze a new finite element method which produce a completely discontinuous piecewice polynomial approximation of convection-diffision equations. This method have optimal order of convergence as classical one.

An outline of the paper is as follows. In the next sections the method is introduced for model problem; existance and uniqueness to the discrete problem is given, and optimal error estimate is obtained for a model problem.

2. Model Problem and Finite Element Approximation

Let Ω be a simply connected polygonal domain of R^2. We consider the model problem : Find u such that

$$\begin{cases} -\Delta u + \beta.\nabla u + \sigma u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{2.1}$$

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where $\beta \in (W^{1,+\infty}(\Omega))^2$, $\sigma \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$. In the sequel we make the assumption

$$\sigma - \frac{1}{2}\,\mathrm{div}\beta \geq \gamma_0 > 0.$$

Assume that we have a regular triangulations $\mathcal{T}_h$ of the domain Ω with triangular finite elements whose diameters are less or equal than h.

First, we introduce the following spaces :

$$W_h = \{v \in L^2(\Omega) : v_{|T} \in H^1(T) \text{ and } \frac{\partial u}{\partial n_T} \in L^2(\partial T), \forall T \in \mathcal{T}_h \text{ and } \ v_{|\partial T \cap \Gamma} = 0$$
$$\text{if } meas(\partial T \cap \Gamma) \neq 0\}$$

where $\dfrac{\partial u}{\partial n_T}$ is the outward normal derivative of the restriction of u to T.

$$V_h = \{v \in L^2(\Omega) : v_{|T} \in P_1(T), \forall T \in \mathcal{T}_h \ \text{ and } \ v_{|\partial T\cap\Gamma} = 0, \text{ if } meas(\partial T \cap \Gamma) \neq 0\}.$$

Let us remark, that we have

$$V_h \subset W_h. \tag{2.2}$$

Finally, we introduce some notation that we will need in the definition and analysis of the finite element approximation of the model problem.

Let E_I be the set of all interior edges and E_T the set of edges of T. For each interior edge l we choose an arbitrary normal direction n and denote the two triangles sharing this edge T_+ and T_- where n points outwards T_+. For a boundary edge l we take n as the outward normal.

We define the jump of $v \in W_h$ on l by

$$[[v]]_l(x) = v_{|T_+}(x) - v_{|T_-}(x), \forall x \in l.$$

For all $T \in \mathcal{T}_h$, we denoted by ∂T^- and ∂T^+ the set defined by :

$$\partial T^- = \{x \in \partial T, \text{ such that } \ \beta.n(x) < 0\}$$

and

$$\partial T^+ = \{x \in \partial T, \text{ such that } \ \beta.n(x) > 0\}$$

where n is the outward normal to T. And we set, for all $v \in W_h$

$$\forall l \in E_I, v^{\mp}(x) = \lim_{\epsilon \longrightarrow 0} v(x \mp \epsilon\beta(x)), \ x \in l,$$

we use the convention $u^{\mp} = 0$ on Γ.

Let $(u, v) \in (W_h)^2 \longrightarrow B_d(u, v)$ the bilinear form defined by

$$\begin{cases} B_d(u,v) = \displaystyle\sum_{T\in\mathcal{T}_h} \int_T \nabla u \nabla v dx + \sum_{l \in E_I} 3\frac{meas(l)}{meas(T_+)} \int_l [[u]]_l [[v]]_l d\sigma \\ \qquad - \displaystyle\sum_{l\in E_I} \int_l \frac{\partial(u_{|T+})}{\partial n} [[v]]_l d\sigma - \alpha \sum_{l \in E_I} \int_l \frac{\partial(v_{|T+})}{\partial n}[[u]]_l d\sigma \end{cases}$$