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TRAPPED MODES AROUND FREELY FLOATING BODIES IN TWO-LAYER FLUIDS

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Abstract. We consider the coupled system modeling the interaction of time-harmonic progressive water-waves with an array of three-dimensional freely floating obstacles in a two-layer fluid. Presenting a variational and operator formulation for the problem and a condition guaranteeing the existence of trapped waves, we give a number of examples of floating structures supporting trapped waves. We also study how the problem parameters (density ratio, obstacle dimensions, layer depths and radian frequency) influence the trapping condition.

Key words. Trapped modes, linear water waves, freely floating obstacle, two-layer fluid.

1. Introduction

The linearized equations modeling the interaction of water waves with freely floating structures were written down over 60 years ago by John [7, 8] (for a more recent presentation see, *e.g.*, Mei *et. al.* [16] and Cal *et. al.* [1]). Since then the behavior of time-harmonic water waves around fixed structures has been quite thoroughly investigated (see Kuznetsov *et al.* [11] and all references therein) but results on freely floating bodies have been much scarcer. Recently though, there has been kind of a revival, see McIver and McIver [15], Porter and Evans [20], Fitzgerald and McIver [5], Kuznetsov [10], Kuznetsov and Motygin [12, 13], Nazarov [17, 18], Nazarov and Videman [19], Cal *et al.* [1], Cal, Dias and Videman [3], and Dias and Videman [4].

This work is directed at coupled free (unforced) oscillations of the freely floating structures with the surrounding unbounded fluid domain. These oscillations, also known as trapped modes, are excluded from John's initial analysis since his interest lay with the conditions for a unique solution thus immediately excluding trapped modes. The corresponding waves are characterized by their propagation in the vicinity of the obstacles that generate them. Studying these motion trapped waves is relevant to offshore activities, think of oil and gas drilling, and for the construction of floating structures such as piers and bridges subject to tides and/or wave motions. Harbor buoys and vessels in channels and fjords are also prone to this kind of oscillations. In the area of energy extraction from waves in the ocean, it has been shown that there are differences in energy efficiency between a large buoy and an array of smaller ones [6].

Our main goal here is to provide examples of (arrays of) floating structures that support trapped modes in a fluid region of finite depth consisting of two horizontally infinite fluid layers of constant density. With our sights set on that goal, we first determine general conditions guaranteeing the existence of waves that propagate

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along a periodic array of freely floating obstacles but decay away from them. Similar analysis was done in the case of fixed obstacles in Cal, Dias and Videman [2], and for freely-floating obstacles in a homogeneous fluid by Dias and Videman [4]. We will adopt the idea, suggested by Nazarov and Videman [19], to separate the motions relevant to buoyancy from the other rigid-body motions of the floating obstacles. As in [19], we also rewrite the equations of motion for the coupled time-harmonic problem as a spectral boundary-value problem consisting of a differential equation and an algebraic system, coupled through boundary conditions. Following the argument presented in [19], we then reduce the original quadratic eigenvalue problem into a linear one.

It follows from Kelvin's Circulation Theorem that an initially irrotational flow cannot stay irrotational if the flow is baroclinic, cf. Kundu *et al.* [9]. One way to approach baroclinicity without losing the advantage of working with velocity potentials is to consider multiple homogeneous fluid layers. Despite being a crude representation of continuous stratification, the multi-layer models are widely used in Geophysical Fluid Dynamics when combined with shallow-water or quasi-geostrophic dynamics.

The paper is organized as follows. First we lay out the equations, divide the domain into periodicity cells by imposing quasi-periodic conditions along the direction of propagation and present the stability conditions. In Section 3 we analyse the problem without obstacles and in Section 4 present the variational and operator formulation. Next we introduce the scheme that reduces the original problem to a linear eigenvalue problem and present general sufficient conditions for the existence of trapped modes. In Section 6 we provide examples of obstacles, surface- or interface-piercing, supporting trapped modes and study the dependence of the trapping and the stability conditions on the problem parameters.

2. Equations of motion

Consider two homogeneous, incompressible, inviscid fluid layers of finite depth lying on top of one another and over a flat bottom. For gravitational stability, assume that the constant density in the lower layer is greater than the one in the upper layer ($\rho_2 > \rho_1 > 0$). The origin of Cartesian coordinates is fixed at the interface between the fluid layers in such a way that the $(x, y)-$plane coincides with its rest position and the $z-$axis points upwards. Partially or totally submerged in the fluid domain, there is a periodic array of obstacles extending to infinity in the y-direction and floating freely under the effect of gravity.

The fluid domain is divided into periodicity cells, infinite in the x-direction, having unit length (after non-dimensionalisation) in the y-direction and containing the same obstacles. The upper and lower fluid layers are denoted by $\Xi^1 = \mathbb{R}^2 \times (0, h_1)$, $h_1 \in \mathbb{R}^+$ and $\Xi^2 = \mathbb{R}^2 \times (-h_2, 0)$, $h_2 \in \mathbb{R}^+$, with h_1 and h_2 being the rescaled layer depths, and the model periodicity cells by

$$\Pi^j = \big\{(x, y, z) \in \Xi^j \,:\, y \in (0,1)\big\}\,, \quad j = 1, 2\,.$$

All length variables have been made non-dimensional by division by the cell length.

Within the periodicity cells, we introduce bounded open sets $\Theta^1 \subset \Pi^1$ and $\Theta^2 \subset \Pi^2$ corresponding to the submerged part of a model obstacle and assume that the fluid regions $\varpi^1 = \Pi^1 \setminus \overline{\Theta^1}$, $\varpi^2 = \Pi^2 \setminus \overline{\Theta^2}$ are Lipschitz domains so that the normal vector is defined almost everywhere on $\partial\varpi^1$ and $\partial\varpi^2$. We also define the