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PROPER ORTHOGONAL DECOMPOSITION CLOSURE MODELS FOR FLUID FLOWS: BURGERS EQUATION

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Abstract. This paper puts forth several closure models for the proper orthogonal decomposition (POD) reduced order modeling of fluid flows. These new closure models, together with other standard closure models, are investigated in the numerical simulation of the Burgers equation. This simplified setting represents just the first step in the investigation of the new closure models. It allows a thorough assessment of the performance of the new models, including a parameter sensitivity study. Two challenging test problems displaying moving shock waves are chosen in the numerical investigation. The closure models and a standard Galerkin POD reduced order model are benchmarked against the fine resolution numerical simulation. Both numerical accuracy and computational efficiency are used to assess the performance of the models.

Key words. Proper orthogonal decomposition (POD), reduced order models (ROMs), closure models for POD, Burgers equation, moving shock wave.

1. Introduction

Proper orthogonal decomposition (POD) is one of the most successful reduced order modeling techniques in dynamical systems. POD has been used to generate reduced order models (ROMs) for the simulation and control of many forced-dissipative nonlinear systems in science and engineering applications [1, 11, 16, 17, 23, 27, 31, 34, 36, 47]. POD extracts the most energetic modes in the system, which are expected to contain the dominant characteristics. The globally supported POD modes are often constructed from high-fidelity numerical solutions (e.g., using finite difference/element/volume methods) and are problem dependent. For many systems, it is possible to obtain a good approximation of their dynamics with few POD modes. The systems built with these POD modes, called POD reduced order models (POD-ROMs) in what follows, are low dimensional and can provide an efficient framework for many applications.

The Galerkin POD-ROM (POD-ROM-G) is the simplest POD-ROM, which results from a Galerkin truncation followed by a projection of the truncated equation onto the space spanned by the POD modes. The POD-ROM-G is an efficient tool for many applications of interest. For fluid flows (see, e.g, [37] and the exquisite survey in [29]), the POD-ROM-G works well for laminar fluid flows. For turbulent flows, however, the POD-ROM-G yields inaccurate results. As carefully explained in [51], for realistic turbulent flows, the high index POD modes that are not included in the POD-ROM-G do have a significant effect on the dynamics of the POD-ROM-G. Several numerical stabilization strategies have been used to address this issue [6, 24, 44]. Building on the analogy with large eddy simulation (LES) [7, 42] (see [39] for alternative approaches), several closure modeling strategies for POD-ROMs of turbulent flows have been proposed over the years [2, 3, 6, 9, 13, 14, 41, 50, 52, 53], starting with the pioneering work in [1]. The main goal of this report is to propose and numerically investigate several closure models for POD-ROMs of fluid flows. Three different classes of closure modeling strategies are considered.

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The first strategy provides additional dissipation to the POD-ROM-G to account for the small scale dissipation effect of the discarded POD modes. The main advantage of this strategy is that it has a negligibly small computational overhead. Different viscosity kernels for POD-ROMs have been suggested in literature (see, e.g., [6, 44]). In this study, these closure models together with three new closure models are presented in a unified framework and their performance is investigated.

The second closure modeling strategy is also of eddy viscosity type. The subgrid-scale operator used to account for the the effects of smaller scales is, however, different from that employed in the first closure modeling strategy. This closure modeling approach is inspired from state-of-the-art LES models, such as the Smagorinksy model [46] or its dynamic counterparts (see, e.g., [19]). The Smagorinsky POD-ROM (POD-ROM-S) has been used in several studies [9, 38, 49, 52, 53]. To the best of our knowledge, the dynamic subgrid-scale POD-ROM was first used in [53]. In this study, we investigate the POD-ROM-S together with one new closure model. In general, these nonlinear closure models have a significant computational overhead (as explained in [52]). For the one-dimensional Burgers equation considered in this study, however, the computational overhead of these two closure models is negligible.

The third closure modeling strategy that we consider is based on the energy conservation concept. This POD-ROM closure modeling approach was introduced by Cazemier [13] (see also [14]). One of the main advantages of this closure model is that it does not require the specification of any free parameter, which is in stark contrast with the closure modeling strategies outlined above. This closure model, however, has a higher computational overhead since it requires the computation of a penalty/drag term. We note, however, that for the one-dimensional Burgers equation considered in this study this computational overhead is negligible.

Overall, there are 10 closure models, both new and current, in the three classes described above. There are numerous other closure modeling strategies for POD-ROM of complex systems (see, e.g., [29, 37, 51]). In general, when these closure models are introduced, they are deemed successful if they satisfy the following two criteria when compared with the fine resolution numerical simulation: (i) the new POD-ROM is relatively accurate; and (ii) the new POD-ROM has a significantly lower computational cost. Given the number of competing closure modeling approaches, a natural *practical* question is which closure model should be used. These intercomparison studies are scarce (see, e.g., [53] for an exception). This study aims at answering the above practical question for the 10 closure models considered herein.

All the 10 POD-ROM closure models considered in this study are investigated in the numerical simulation of the Burgers equation. We emphasize again that these closure models are developed for POD-ROMs of realistic turbulent flows. In order to thoroughly assess their performance, however, as a first step, we consider the one-dimensional Burgers equation displaying challenging moving shock waves. This simplified setting allows us to carefully assess the performance of the 10 closure models considered in this study and also carry out a parameter sensitivity study. Of course, once we get a better understanding of the performance and limitations of these closure models, we will investigate them in realistic turbulent flow settings, such as those considered in [53]. We also note that this progressive evaluation is common in the POD-ROM literature [9, 28] or in the turbulence modeling literature [4, 5, 10, 18, 20, 26, 33].