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CONTINUOUS SEDIMENTATION IN CLARIFIER-THICKENER UNITS: MODELING MACROSCOPIC CONSERVATION LAWS FROM MICROSCOPIC INTERACTING PARTICLES

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Abstract. We study a model of continuous sedimentation. Under idealizing assumptions, the settling of the solid particles under the influence of gravity can be described by the initial value problem for a one-dimensional scalar conservation law with a flux function that depends discontinuously on the spatial position.
There exists several entropy conditions related to the same conservation law in the literature giving rise to uniqueness. The same initial data may give rise to different entropy solutions, depending on the criteria one selects. This motivated us to derive the PDE together with an entropy condition as a hydrodynamic limit from a microscopic interacting particle system.We are inclined to prefer the entropy solution selected by this method. It turns out that this is an entropy condition suggested by Audusse and Pethame in a different context.

Key words. hyperbolic conservation laws, discontinuous flux functions, hydrodynamic limits, microscopic particle systems

1. Modeling a clarifyer-thickener model

Given a suspension of small solid particles dispersed in a fluid in a box. Under gravity these particles settle to the bottom of the box. The suspension shall be modeled by a mixture of two superimposed continuous media. Let v_s be the velocity of the solid phase, v_f the velocity of the fluid phase, and ρ the local solid concentration. The equations of continuity give:

$$\begin{aligned} u_t + (uv_s)_x &= 0 \\ u_t - ((1-u)v_f)_x &= 0 \end{aligned}$$

Now we introduce a volume average velocity

$$q := uv_s + (1-u)v_f$$

This allows the above two conservation of mass equations to be combined to a single scalar equation of the type:

$$q_t + g(x,t,q)_x = 0 \qquad x \in R, t > 0$$

. Here the flux function in general depends in a discontinuous fashion on the space variable x. For more details see [3].

2. Scalar conservation laws with discontinuous flux

We are concerned with the following class of scalar conservation laws, describing flow through porous media

$$\begin{cases} \partial_t \rho(t,x) + \partial_x F(x, \rho(t,x)) = 0 \\ \\ \rho|_{t=0} = \rho_0(x) \end{cases} \tag{2.1}$$

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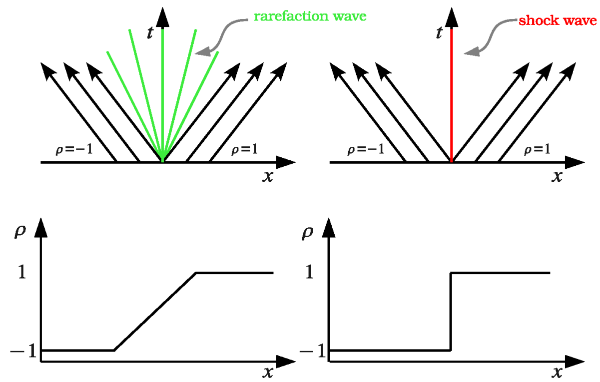

FIGURE 1

where $F(\cdot,\rho)$ is continuous except on a set of measure zero. This discontinuity may appear for example, if there is a sudden change of the porosity.

Recall that even if the space dependency of the flux is continuous, weak solutions of this partial differential equation may not be unique. The most known example of this is probably the inviscid Burger equation which reads as

$$\begin{cases} \partial_t\rho(t,x) + \frac{1}{2}\partial_x\rho^2(t,x) = 0 \\ \rho_0(x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases} \end{cases}$$

This equation may allow two solutions. The first solution produces a rarefaction at the origin, the second solution produces a shock at the origin:

Therefore the solutions have to be considered together with an admissibility criteria called entropy condition which picks a unique solution. The hope is, that the entropy condition naturally leads to the physical relevant one.

If the flux function $F(\cdot,\rho)$ is a smooth function in space, the equation is well understood. In this case Kruzkov in [10] proposed an entropy inequality for which uniqueness and existence have been shown. The entropy inequality, in the weak sense, reads as

$$\partial_t |\rho - c| + \partial_x \{\text{sign}(\rho - c)\,(F(x,\rho) - F(x,c))\} + \text{sign}(\rho - c)\partial_x F(x,c) \leq 0,$$

for each constant c. Obviously, if there are discontinuities in the space dependency of the flux function F, the third term of the last expression does not make sense and hence the Kruzkov inequality does not hold anymore. For this case several admissibility criteria have been proposed in the literature, for example see [6, 9, 7], for which uniqueness and existence have been shown within the various classes. In [1] the authors proved uniqueness of a Kruzkov-type entropy inequality, but not existence. Unfortunately these entropy conditions are in general not equivalent, that means, it may happen that one admissibility criteria selects a unique entropy solution, which is different from the unique entropy solution selected by an other admissibility criteria.