INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING, *SERIES B*
Volume 4, Number 4, Pages 353–371

ON A GALERKIN DISCRETIZATION OF 4TH ORDER IN SPACE AND TIME APPLIED TO THE HEAT EQUATION

SHAFQAT HUSSAIN, FRIEDHELM SCHIEWECK, STEFAN TUREK, AND PETER ZAJAC

Abstract. We present a new time discretization scheme based on the continuous Galerkin Petrov method of polynomial order 3 (cGP(3)-method) which is combined with a reduced numerical time integration (3-point Gauß-Lobatto formula). The solution of the new approach can be computed from the solution of the lower order cGP(2)-method, which requires to solve a coupled 2×2 block system on each time interval, followed by a simple post-processing step, such that we get the higher accuracy of 4th order in time in the standard L^2-norm with nearly the cost of the cGP(2)-method. Moreover, the difference of both solutions can be used as an indicator for the approximation error in time. For the approximation in space we use the nonparametric $\widetilde{Q}_3$-element which belongs to a family of recently derived higher order nonconforming finite element spaces and leads to an approximation error in space of order 4, too, in the L^2-norm. The expected optimal accuracy of the full discretization error in the L^2-norm of 4th order in space and time is confirmed by several numerical tests. We discuss implementation aspects of the time discretization as well as efficient multigrid methods for solving the resulting block systems which lead to convergence rates being almost independent of the mesh size and the time step. In our numerical experiments we compare different higher order spatial and temporal discretization approaches with respect to accuracy and computational cost.

Key words. continuous Galerkin-Petrov method, nonconforming FEM, heat equation, multigrid method

1. Introduction

Regarding the 'optimal complexity' of time-dependent PDE simulations, as for instance for the heat equation as simplest example, discretization methods of order 4 in space and time seem to be excellent candidates, particularly in 3D. Since during grid refinement each regular refinement step in time (factor 2) and in space (factor $8 = 2^3$) leads to 16 times more unknowns, a 4th order scheme is necessary to balance this increase in numerical complexity (while for 2D problems, a refinement in space-time leads to 8 times more unknowns such that a 3rd order space-time scheme is already sufficient). Hereby we assume that a specifically adapted multigrid solver is available which exhibits linear complexity w.r.t. the problem size.

A candidate for such a higher order temporal discretization scheme will be proposed in this paper. This scheme arises if we apply in the usual cGP(3)-method [4] (with a cubic polynomial ansatz in time) a reduced numerical time integration by means of the 3-point Gauß-Lobatto formula. The idea has been previously published in the preprint [8]. We call the resulting scheme the "cGP-C1(3)-method" since it turns out that the numerical solution is a C^1-function in time. In numerical experiments we show that the discretization error in the L^2-norm is of order 4 with respect to the time step size in the whole time interval whereas the standard cGP(2)-method (see [4, 10]) is only superconvergent of order 4 at the endpoints of

Received by the editors October 5, 2012 and, in revised form, August 15, 2013.
2000 *Mathematics Subject Classification.* 65M12, 65M55, 65M60 .
The authors want to express their gratitude to the German Research Association (DFG) and the Higher Education Commission (HEC) of Pakistan for their financial support of the study: SCHI 576/2-1, TU 102/35-1 and LC06052 by MSMT.

the time intervals. A theoretical proof of the 4th order accuracy in the case of a nonlinear ODE system has been presented in [8].

Concerning the numerical cost, it turns out that the numerical solution of the cGP-C1(3)-method can be computed on each time interval by solving the (same) coupled 2×2 block system in space of the cGP(2)-method followed by a simple post-processing step which requires to solve one linear system with the mass matrix. Moreover, we demonstrate numerically that the L^2-norm of the difference between the cGP(2) and cGP-C1(3)-solution can be used as indicator for the approximation error in time of the cGP(2)-solution.

As corresponding 4th order space discretization, we use the standard Galerkin Finite Element Method (FEM) with higher order nonconforming (nonparametric) quadrilateral $\widetilde{Q}_3$-elements (see [5]). We use such nonconforming elements since they show an advantageous numerical behaviour for saddle-point problems, particularly for incompressible flow problems together with discontinuous pressure approximations, and they are preferable for parallel computing due to the fact that they only require edge- or face-oriented communication which simplifies the parallel data exchange. To solve the associated linear (block) systems, we propose a geometrical multigrid solver with canonical grid transfer operators due to the FEM space $\widetilde{Q}_3$. The numerical experiments confirm that such multigrid methods [2, 5, 6, 11] are very efficient solvers since their rate of convergence is almost independent of the space mesh size and the size of the time step on structured as well as on semi-structured meshes.

Note that our proposed method is of 4th order accurate only if the exact solution is smooth enough, i.e., if its 4th order derivatives with respect to time and space are bounded. If the exact solution is less regular then the accuracy of our method will be reduced to the order of smoothness of the solution. We compare in numerical experiments, where the exact solution is known and sufficiently smooth, our new space-time discretization of order 4 with two other discretizations of order 3 and 2 concerning the achieved accuracy in relation to the required CPU-time. The results clearly show the big advantage of the developed high order methodology and the superior computational complexity during grid refinement.

2. The cGP(k)-method for the heat equation

As a model problem we consider the heat equation: *Find* $u : \Omega \times [0,T] \to \mathbb{R}$ *such that*

$$
\begin{array}{rcll}
d_t u - \triangle u & = & f & \text{in} \quad \Omega \times (0,T), \\
u & = & 0 & \text{on} \quad \partial\Omega \times [0,T], \\
u(x,0) & = & u_0(x) & \text{for} \quad x \in \Omega,
\end{array} \tag{1}
$$

where $u(x,t)$ denotes the temperature in the point $x \in \Omega$ at time $t \in [0,T]$, $f : \Omega \times (0,T) \to \mathbb{R}$ a given source term and $u_0 : \Omega \to \mathbb{R}$ the initial temperature field at time $t = 0$. For simplicity, we assume a polygonal domain $\Omega \subset \mathbb{R}^2$ and homogeneous Dirichlet boundary conditions. Then, problem (1) can be considered as an evolution problem in the Hilbert space $V := H_0^1(\Omega)$. Let $(\cdot,\cdot)_\Omega$ denote the inner product in $L^2(\Omega)$ and $a(\cdot,\cdot)$ the following bilinear form

$$a(u,v) := (\nabla u, \nabla v)_\Omega \qquad \forall\, u, v \in V.$$

For the time discretization, we decompose the time interval $I = [0,T]$ into subintervals $I_n := [t_{n-1}, t_n]$, $n = 1, \ldots, N$. Applying the *exact cGP(k)-method* (see [4] for details) we get a time marching process with the following "I_n-problem": *Find*