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FINITE ELEMENT METHOD FOR AMERICAN OPTION PRICING: A PENALTY APPROACH

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Abstract. The model for pricing of American option gives rise to a parabolic variational inequality. We first use penalty function approach to reformulate it as an equality problem. Since the problem is defined on an unbounded domain, we truncate it to a bounded domain and discuss error due to truncation and penalization. Finite element method is then applied to the penalized problem on the truncated domain. By coupling the penalty parameter and the discretization parameters, error estimates are established when the initial data in H_0^1. Finally, some numerical experiments are conducted to confirm the theoretical findings.

Key words. Penalty method, American options, variational inequality, finite element method, error analysis.

1. Introduction

In a financial market, an option is a contract which gives to its owner the right to buy (call option) or to sale (put option) a fixed quantity of assets of a specified stock at a fixed price (exercise or strike price) on (European option) or before (American option) a given date called expiry date. It is known that price of an American option is governed by a linear complementarity problem [5, 7, 16] involving the Black-Scholes differential operator and a constraint (or obstacle) on the value of the option.

In literature, there are several methods for the valuation of European and American options. The first numerical approach to Black-Scholes equation is a lattice method proposed in [2]. Since, it is a linear complementarity problem, finite difference methods need to combine with other techniques for solving discrete linear complementarity problem using methods like PSOR algorithm [18], operator splitting [6]. Approaching linear complementarity problem using penalty method is not quite new, for example, penalty method and front fixing method together with finite difference method are discussed in [17, 11, 12]. Finite difference methods are by far the simplest and been favourite in computational finance but it is not suitable for mesh adaptivity.

Finite volume methods are also used for pricing American/European option with constant or time-dependent volatility. Wang *et al.* [14, 15] have proposed a fitted finite volume method for spatial discretization and an implicit time stepping technique for temporal discretization which is combined with power penalty method for option pricing. The analysis is performed within the framework of the vertical method of lines, where the spatial discretization is formulated as a Petrov-Galerkin finite element method with each basis function of the trial space being determined by a set of two-point boundary value problems.

Finite element methods seem at first glance unnecessarily more complex than finite difference scheme for finance, where a large class of problems is one dimensional in space. However, these methods are very flexible for mesh adaptivity. Earlier,

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authors in [1], Allegretto *et. al.* have considered American options and reformulated it as variational inequalities of the heat equation with nonlocal boundary conditions on a bounded domain. Approximations of the variational inequality is discussed by using piecewise linear finite elements and the backward Euler scheme. They have established existence, uniqueness, and stability of the discrete solutions and derived error estimates of order $O(k^{1/2}+h)$ in $\ell^2(H^1)$. In the context of finite element methods applied to parabolic variational inequalities, Scholz [13] has discussed finite element method for space and backward Euler method in time combined with penalty approach and obtained $O((h+\epsilon^{-1/2}h^2)+k(1+\epsilon^{-1/2}))$ error estimates in $\ell^2(H^1)$-norm, when the initial data is in H^2.

In this article, we discus error estimates in $L^\infty(L^2)$, $L^2(L^2)$ and $L^2(H^1)$-norms using finite element method combined with penalty as well as truncation approach when the initial condition is in H^1 only.

In Section 2, we first truncate the domain from $\mathbb{R}$ to finite bounded domain and then problem is formulated as a semilinear parabolic partial differential equation using penalty function. Section 3 deals with truncation error of penalized problem. *A priori* bounds are discussed for the penalized problem and convergence of penalized solution to variational inequality problem are derived in Section 4. In Section 5, finite element method is applied to the penalized problem and error estimates in $L^2(L^2)$ and $L^2(H^1)$-norms are established for semi-discrete approximation. In Section 6, a completely discrete scheme based on backward Euler method is analysed and error estimates are discussed in $\ell^2(L^2)$ and $\ell^2(H^1)$ norms. Finally, some numerical experiments which confirm our theoretical findings in Section 7.

2. American options and penalty formulation

In the framework of the Black-Scholes model [16], we now assume that $S(t)$, the price of the underlying risky asset follows a geometric Brownian motion with volatility $\sigma > 0$ and interest rate $r > 0$. An American put option with strike price K and maturity time T gives the holder a right to sell the underlying asset at any time t, on and before expiration date say T at a price K. Let V denote the value of an American put option with strike price K and expiry date T and let S_t denote the price of the underlying asset at time t. Under the standard assumption of a frictionless market without arbitrage, one finds that the solution $V(S,t)$ satisfies the following parabolic variational inequality problem, see [16]:

$$\begin{aligned} LV := -\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2(t)S^2\frac{\partial^2 V}{\partial S^2} - r(t)S\frac{\partial V}{\partial S} + r(t)V \geq 0 \; a.e. \; in \; \mathbb{R}_+, 0 \leq t < T \\ V(S,t) - V^*(S) \geq 0 \; a.e. \; in \; \mathbb{R}_+, 0 \leq t < T \\ LV(S,t)\cdot[V(S,t) - V^*(S)] = 0 \; a.e. \; in \; \mathbb{R}, 0 \leq t < T \end{aligned}$$

where the volatility of asset $\sigma(t) > 0$, the interest rate $r(t) \geq 0$, and V^* is the final (payoff) condition defined by

$$V(S,T) = V^*(S) = \max\{K - S, 0\}. \tag{1}$$

Using transformation from t to $T-t$ and $x = \log S$, then the function $u(x,t) := V(e^x, T-t)$ satisfies the following:

$$\begin{aligned} \frac{\partial u}{\partial t} + \mathcal{L}u \geq 0, \quad x \in \mathbb{R}, 0 < t \leq T, \\ u(x,t) \geq u^*(x), \; \left(\frac{\partial u}{\partial t} + \mathcal{L}u\right)\cdot(u - u^*) = 0 \;\; x \in \mathbb{R}, 0 < t \leq T \\ u(x,0) = u_0(x) = \max(K - e^x, 0), \; x \in \mathbb{R}, \end{aligned}$$