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DYNAMIC INSTABILITY OF STATIONARY SOLUTIONS TO THE NONLINEAR VLASOV EQUATIONS

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Abstract. We present the dynamic instability of smooth compactly supported stationary solutions to the nonlinear Vlasov equations with self-consistent attractive forces. For this, we explicitly construct a one-parameter family of perturbed solutions via the method of the Galilean boost. Initially, these perturbations can be close to the given stationary solution as much as possible in any L^p-norm, $p \in [1, \infty]$, and have the same local mass density profile as a stationary solution, but a different bulk velocity profile. At the macroscopic level, these perturbations correspond to the traveling waves with compact supports. However in finite-time, the phase-space supports of these perturbations will be disjoint from the support of the given stationary solution. This establishes the dynamic instability of stationary solutions in any L^p-norm.

Key words. Dynamic instability, Galilean boost, stationary solution, Vlasov-Poisson system, Vlasov-Yukawa system, Euler-Poisson system

1. Introduction

The purpose of this paper is to present a dynamic instability of smooth L^p-stationary waves to nonlinear Vlasov equations with radially symmetric force potential. Consider an ensemble of many interacting particles through a self-consistent attractive conservative force. For definiteness, we assume that the force potential U is generated collectively by the convolution between the local mass density $\rho = \rho(x, t)$ and a spherically symmetric kernel $K = K(|x|)$. Let $f = f(x, v, t)$ be a one-particle distribution function at the phase-coordinate $(x, v) \in \mathbb{R}^6$ at time $t \geq 0$. The spatial-temporal evolution of the distribution function f is governed by the self-consistent nonlinear Vlasov equation:

$$
\begin{aligned}
&\partial_t f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0, \quad x, v \in \mathbb{R}^3,\ t > 0, \\
&U = K *_x \rho, \quad \rho = \int_{\mathbb{R}^3} f dv,
\end{aligned}
\tag{1.1}
$$

subject to initial datum:

$$
f(x, v, 0) = f^{in}(x, v). \tag{1.2}
$$

The prototypical examples of (1.1) are the Vlasov-Poisson(in short V-P) system and the Vlasov-Yukawa(V-Y) system. The nonlinear Vlasov equations have many physical and engineering applications in the modeling of an electron gun, plasma sheath and galaxies as a large ensemble of stars in plasma physics and astrophysics [4, 16]. The Cauchy problem for (1.1) can be treated via the Bardos-Degond approach [1] for small data, Pffaffelmoser's characteristic method [18] and Lions-Perthame's velocity moment arguments [17] for large data. For many interesting issues on the weak solutions, stability and dispersion estimates, we refer to a recent survey article

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[19].

We next briefly discuss main motivation of this work. In [5], Chae and Ha investigated the uniform L^1-stability of the V-P system using the dispersion estimate and Gronwall's estimate in the class of Bardos-Degond solutions. More precisely, for any two small and decaying smooth solutions f and g with corresponding initial data f^{in} and g^{in}, they derived a Growl's inequality for $||f(t) - g(t)||_{L^1(\mathbb{R}^{2d})}$:

$$\begin{aligned} &||f(t) - g(t)||_{L^1(\mathbb{R}^{2d})} \\ &\qquad \le ||f^{in} - g^{in}||_{L^1(\mathbb{R}^{2d})} + C\int_0^t (1+s)^{-(d-2)}||f(s) - g(s)||_{L^1(\mathbb{R}^{2d})}ds. \end{aligned} \tag{1.3}$$

Note that for a high dimension with $d \ge 4$, the time-factor $(1+s)^{-(d-2)}$ inside the integrand is integrable, hence the relation (1.3) results in the uniform L^1-stability with respect to initial data:

$$||f(t) - g(t)||_{L^1(\mathbb{R}^{2d})} \le C||f^{in} - g^{in}||_{L^1(\mathbb{R}^{2d})}, \qquad t \ge 0, \tag{1.4}$$

where the positive constant C appearing in R.H.S. is independent of t. However for $d = 3$, we have a non-integrable factor $(1+s)^{-1}$ in (1.3). Hence we have

$$||f(t) - g(t)||_{L^1(\mathbb{R}^6)} \le (1+t)^C||f^{in} - g^{in}||_{L^1(\mathbb{R}^6)}. \tag{1.5}$$

Therefore the direct stability estimate based on the dispersion estimate and Gronwall's inequality is inconclusive for the uniform L^1-stability (1.4) in physically interesting three dimensions. It still remains as an interesting open problem. Recently authors obtained a negative clue for the possible scenario on the uniform L^1-stability of the V-P system in three dimensions, in particular, they showed that the non-existence of the asymptotic completeness for the V-P system in three dimensions in [7], i.e., the corresponding linear transport flow

$$\partial_t f + v \cdot \nabla_x f = 0$$

cannot be used as an approximate flow for the nonlinear dynamics of (1.1) in a large-time regime. This might suggest a possible scenario of L^1-instability of small solutions to the Vlasov-Poisson system in three dimensions. Of course, this paper do not resolve this issue completely, but our result in this paper suggests that in general, the uniform L^1-stability is not true for smooth solutions(see Remark 1.1). Therefore, suitable smallness assumption on initial data is crucially needed, if we want to have the uniform L^1-stability of the V-P system. In contrast, for some regularized V-P systems such as the V-Y system and the Vlasov-Poisson-Fokker-Planck system, the uniform L^1-stability was obtained for small and decaying solutions in three dimensions [12, 13]. Hence the V-P system in three dimensions lies on the border line from the viewpoint of L^1-stability among self-consistent nonlinear Vlasov equations.

The main result of this paper is the L^p-instability of smooth stationary solution with a compact support.

Theorem 1.1. *Let $f_0 = f_0(x, v)$ be a $\mathcal{C}_c^1$-stationary solution to system* (1.1). *Then for any $\varepsilon > 0$ and $p \in [1, \infty]$, there exists a smooth perturbation f^{in} of f_0 and $T = T(\varepsilon) > 0$ such that the smooth solution $f = f(t)$ with initial datum f^{in} satisfies*

$$||f^{in} - f_0||_{L^p(\mathbb{R}^6)} < \varepsilon \quad \textit{and} \quad ||f(t) - f_0||_{L^p(\mathbb{R}^6)} = 2^{1/p}||f_0||_{L^p(\mathbb{R}^6)}, \quad t \ge T = T(\varepsilon).$$