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A NUMERICAL STUDY FOR A VELOCITY-VORTICITY-HELICITY FORMULATION OF THE 3D TIME-DEPENDENT NSE

KEITH J. GALVIN, HYESUK K. LEE, AND LEO G. REBHOLZ

Abstract. We study a finite element method for the 3D Navier-Stokes equations in velocity-vorticity-helicity formulation, which solves directly for velocity, vorticity, Bernoulli pressure and helical density. Moreover, the algorithm strongly enforces solenoidal constraints on both the velocity (to enforce the *physical* law for conservation of mass) and vorticity (to enforce the *mathematical* law that div(curl)= 0). We prove unconditional stability of the velocity, and with the use of a (consistent) penalty term on the difference between the computed vorticity and curl of the computed velocity, we are also able to prove unconditional stability of the vorticity in a weaker norm. Numerical experiments are given that confirm expected convergence rates, and test the method on a benchmark problem.

Key words. Navier-Stokes equations, Finite element method, Velocity-Vorticity-Helicity formulation.

1. Introduction

Approximating solutions to the 3D Navier-Stokes equations (NSE) is an important subtask in many engineering applications, and improvements in methods to do so remains an active and important area of research. To this end, we study an accurate and efficient numerical method for approximating solutions to the incompressible NSE that is based on a natural 2-step linearization of a finite element discretization of the velocity-vorticity-helicity (VVH) formulation, together with an element choice and mesh condition that leads to efficient and optimally accurate solves of the resulting saddle point systems as well as strong enforcement of the solenoidal constraints on the velocity and vorticity. Recent work by the authors has shown in [16] that the incompressible Navier-Stokes system in a bounded, connected domain $\Omega \subset \mathbb{R}^3$, with a piecewise smooth boundary $\partial\Omega$, end time T, and force field $\mathbf{f} : Q \to \mathbb{R}^3$ (where $Q := (0,T) \times \Omega$), can be equivalently written in VVH form as:

Find $\mathbf{u} : Q \to \mathbb{R}^3, p : Q \to \mathbb{R}$ satisfying

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \mathbf{w} \times \mathbf{u} + \nabla P = \mathbf{f} \quad \text{in} \quad Q, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad \overline{Q}, \tag{1.2}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{in} \quad \Omega, \tag{1.3}$$

$$\mathbf{u} = \boldsymbol{\phi} \quad \text{on} \quad \partial\Omega \times (0,T), \tag{1.4}$$

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and find $\mathbf{w} : Q \to \mathbb{R}^3, \eta : Q \to \mathbb{R}$ satisfying

$$\frac{\partial \mathbf{w}}{\partial t} - \nu\Delta\mathbf{w} + 2\mathbf{D}(\mathbf{w})\mathbf{u} - \nabla\eta = \operatorname{curl}\mathbf{f} \quad \text{in} \quad Q, \tag{1.5}$$

$$\nabla \cdot \mathbf{w} = 0 \quad \text{in} \quad \overline{Q}, \tag{1.6}$$

$$\mathbf{w}|_{t=0} = \operatorname{curl}\mathbf{u}_0 \quad \text{in} \quad \Omega, \tag{1.7}$$

$$\mathbf{w} = \operatorname{curl}\mathbf{u} \quad \text{on} \quad (0,T)\times\partial\Omega, \tag{1.8}$$

where $\mathbf{u}$ represents the velocity, P is the Bernoulli pressure, $\mathbf{f}$ an external force, $\nu > 0$ the kinematic viscosity coefficient, $\boldsymbol{\phi}$ is a Dirichlet boundary condition for velocity satisfying $\int_\Omega \boldsymbol{\phi}\cdot\mathbf{n} = 0\ \forall t \in (0,T)$, $\mathbf{w} := \operatorname{curl}\mathbf{u}$ is the vorticity, $\eta := \mathbf{u}\cdot\mathbf{w}$ is the helical density, and $\mathbf{D}(\mathbf{w}) := \frac{1}{2}(\nabla\mathbf{w} + (\nabla\mathbf{w})^T)$ the symmetric part of the vorticity gradient. Since its initial development, VVH has been studied in several applications, including numerical methods for solving steady incompressible flow [11], the Boussinesq equations [15], and as a selection criterion for the filtering radius in the NS-$\overline{\omega}$ turbulence model [12], all with excellent results. The work herein will extend numerical study of VVH to the time-dependent case.

The VVH formulation has four important characteristics that make it attractive for use in simulations. First, numerical methods based on finding velocity and vorticity tend to be more accurate (usually for an added cost, but not necessarily with VVH) [20, 21, 18, 19, 13], and especially in the boundary layer [6]. Second, it solves directly for the helical density η, which may give insight into the mysterious quantity helicity, $H = \int_\Omega \eta\ d\mathbf{x}$, which is not well understood but is believed to play a fundamental role in turbulence [1, 14, 9, 2, 3, 4, 8, 7]. VVH is the first formulation to directly solve for this helical quantity. Third, the use of $\nabla\eta$ in the vorticity equation enables η to act as a Lagrange multiplier corresponding to the divergence free constraint for the vorticity, analogous to how the pressure relates to the conservation of mass equation. VVH is the first velocity-vorticity method to naturally enforce the incompressibility of the vorticity. Finally, the structure of the VVH system allows for a natural splitting of the system into a 2-step linearization, since freezing vorticity in the velocity equation linearizes the equation, and similarly freezing velocity in the vorticity equation linearizes this equation as well. A numerical method based on such a splitting was proposed in [16], and when coupled with a finite element discretization, was shown to be accurate on some simple test problems. In the present study, we will study further this discretization of VVH (defined precisely in Section 3), by providing a rigorous stability analysis (including for the vorticity), testing it with different boundary conditions for the vorticity, and testing the method on benchmark problems.

This paper is arranged as follows. In Section 2, we provide the necessary notation and preliminaries to allow for a smooth analysis to follow. In Section 3, we present the 2-step method, and provide a stability analysis for it. Lastly, in Section 4, we test the proposed method on benchmark problems.

2. Preliminaries

We present now preliminary results and notation, for the function spaces to be used, and describe the discrete setting.

2.1. Function spaces.

We will use bold font to denote vector function spaces,

$$\mathbf{H}^1(\Omega) := (H^1(\Omega))^3, \quad \mathbf{H}^1_0(\Omega) := (H^1_0(\Omega))^3,$$