INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING, *SERIES B*
Volume 2, Number 2-3, Pages 262–279

SPLINE INTERPOLATION OF FUNCTIONS WITH A BOUNDARY LAYER COMPONENT

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Abstract. This paper is concerned with the spline interpolation of functions with a boundary layer component. It is shown that polynomial formulas of spline interpolation for such functions lead to significant errors. We proposed nonpolynomial spline interpolation formulas, fitted to a boundary layer component. Nonsmooth and smooth on whole interval interpolants are constructed. The accuracy of constructed formulas is estimated. Numerical results are discussed.

Key words. Function, boundary layer, large gradients, nonpolynomial spline interpolation, uniform accuracy.

1. Introduction

Methods of the spline interpolation for functions with bounded derivatives are well known, see e.g. [1], [15]. But when we apply polynomial interpolation methods to functions with large gradients, it leads to significant errors. In this article we consider an interpolation problem for functions with a boundary layer component. We construct interpolation formulas fitted to a boundary layer component and prove that proposed formulas have uniformly small interpolation error on any uniform mesh. Then we construct difference formulas for a derivative on the base of proposed interpolations.

Let the function $u(x)$ be smooth enough and has the representation

$$u(x) = p(x) + \gamma\Phi(x), \quad x \in [0,1], \tag{1.1}$$

where $\Phi(x)$ is known function with regions of large gradients and the function $p(x)$ is the regular part of $u(x)$, bounded together with some derivatives, the constant γ is unknown. The representation (1.1) holds for solutions of problems with boundary layers [6].

The purpose of the article is to develop spline interpolation technique on a uniform mesh to functions with a boundary layer component.

Let the function $u(x)$ be given at nodes of the uniform mesh Ω :

$$\Omega = \{x_n : x_n = nh,\ n = 0,1,\dots,N,\ Nh = 1\},\ \Delta_n = [x_{n-1}, x_n].$$

We denote $u_n = u(x_n),\ n = 0,1,2,\dots,N$.

At first, we show that there is a necessity to construct special interpolations for the functions of the form (1.1). We consider the linear interpolation formula

$$u_2(x) = (u_n - u_{n-1})\frac{x - x_n}{h} + u_n,\ x \in \Delta_n. \tag{1.2}$$

It is known [15] that the next estimate holds

$$|u_2(x) - u(x)| \le \frac{h^2}{8}\max_{s\in\Delta_n}|u''(s)|,\ x \in \Delta_n. \tag{1.3}$$

Received by the editors July 3, 2011.
2000 *Mathematics Subject Classification.* 65L10, 65N06, 65N12.
Supported by Russian Foundation for Basic Research under Grants 10-01-00726, 11-01-00875.

According to (1.3), we have the second order accuracy if the derivative $u''(x)$ is uniformly bounded. Consider the function $u(x) = \exp(-\varepsilon^{-1}x),\ \varepsilon > 0$. Then the derivative $u''(0) = \varepsilon^{-2}$ is not bounded, if ε tends to zero. Let $\varepsilon = h$, then

$$u_2(h/2) - u(h/2) \approx 0,0774$$

for any step h. The interpolation error can not decrease, when $h \to 0$. So, we need interpolation formulas with the property of uniform accuracy for functions of the form (1.1).

To illustrate that there exist functions with the representation (1.1), we consider the problem

$$\varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), \quad u(0) = A,\ u(1) = B, \tag{1.4}$$

where

$$a(x) \geq \alpha > 0, b(x) \geq 0,\ \varepsilon > 0,$$

and the functions $a(x), b(x), f(x)$ are smooth enough. According to [5], the solution of problem (1.4) has exponential boundary layer near the point $x = 0$ and the representation (1.1) for $u(x)$ holds with

$$\Phi(x) = \exp(-a_0\varepsilon^{-1}x) \tag{1.5}$$

and

$$\left|p^{(j)}(x)\right| \leq C_0\left[\varepsilon^{1-j}\exp(-\alpha\varepsilon^{-1}x) + 1\right],\ 0 \leq j \leq 4,$$

where $a_0 = a(0),\ \gamma = -\varepsilon u'(0)/a_0,\quad |\gamma| \leq C_1$. Note that

$$\Phi'(x) < 0,\ \Phi''(x) > 0,\ x \in [0,1].$$

Notations. We mean that $C, C_i,\ i \geq 0$, are some positive constants independent of the function $\Phi(x)$, its derivatives and h. In the case of exponential boundary layer it corresponds to the requirement that C, C_i do not depend on the parameters ε and h.

2. Fitted two-point spline interpolation

We will construct the spline interpolant $u_{\Phi,2}(x)$, taking into account the conditions

$$u_{\Phi,2}(x_{n-1}) = u_{n-1}, \quad u_{\Phi,2}(x_n) = u_n \tag{2.1}$$

for each mesh interval Δ_n. We seek $u_{\Phi,2}(x)$ in the form

$$u_{\Phi,2}(x) = M_1 + M_2\Phi(x).$$

Using conditions (2.1), we obtain

$$u_{\Phi,2}(x) = \frac{u_n - u_{n-1}}{\Phi_n - \Phi_{n-1}}[\Phi(x) - \Phi_n] + u_n,\ x \in \Delta_n, \tag{2.2}$$

where $\Phi_n = \Phi(x_n)$. The interpolant (2.2) is exact for the boundary layer component $\Phi(x)$. We assume that the function $\Phi(x)$ is strictly monotone in each interval Δ_n, then the relation (2.2) is correct. The error of interpolation (2.2) was estimated in [13], where next estimates were obtained

$$|u_{\Phi,2}(x) - u(x)| \leq 2\max_{s\in\Delta_n}|p'(s)|\,h,\ x \in \Delta_n, \tag{2.3}$$

$$|u_\Phi(x) - u(x)| \leq \Big[\max_{s\in\Delta_n}|p''(s)| + \max_{s\in\Delta_n}|p''_{\Phi,2}(s)|\Big]\frac{h^2}{8},\ x \in \Delta_n. \tag{2.4}$$

Here $p_{\Phi,2}(x)$ is the interpolant (2.2) for the function $p(x)$.