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EXACT FINITE DIFFERENCE SCHEMES FOR SOLVING HELMHOLTZ EQUATION AT ANY WAVENUMBER

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Abstract. In this study, we consider new finite difference schemes for solving the Helmholtz equation. Novel difference schemes which do not introduce truncation error are presented, consequently the exact solution for the Helmholtz equation can be computed numerically. The most important features of the new schemes are that while the resulting linear system has the same simple structure as those derived from the standard central difference method, the technique is capable of solving Helmholtz equation at any wavenumber without using a fine mesh. The proof of the uniqueness for the discretized Helmholtz equation is reported. The power of this technique is illustrated by comparing numerical solutions for solving one- and two-dimensional Helmholtz equations using the standard second-order central finite difference and the novel finite difference schemes.

Key words. Helmholtz equation, wavenumber, radiation boundary condition, finite difference schemes, exact numerical solution.

1. Introduction

The study of wave phenomena is important in many areas of science and engineering. The Helmholtz equation arises from time-harmonic wave propagation, and the solutions are frequently required in many applications such as aero-acoustic, underwater acoustics, electromagnetic wave scattering, and geophysical problems. Finite difference methods are commonly used to solve the Helmholtz equation. In addition to the standard central finite difference, Sutmann [16] derived a new compact finite difference scheme of sixth order for the Helmholtz equation and the convergence characteristics and accuracy were compared for a broad range of wavenumbers. Accurate high order finite difference methods were reported in Singer and Turkel [14, 15], Harari and Turkel [10]. A new nine-point sixth-order accurate compact finite-difference method for solving the Helmholtz equation in one and two dimensions was developed and analyzed in [13]. Other numerical techniques such as finite element and spectral methods have been applied to solve the problem. Babuška and Ihlenburg [11] used the h-version of the finite element method with piecewise linear approximation to solve a one-dimensional model problem, Babuška et al. [3] presented a systematic analysis of a posteriori estimation for finite element solutions . Harari and Magoulés [9] considered the Least-Squares stabilization of finite element computation for the Helmholtz equation. Babuška and Sauter [4] found that the solution of the Galerkin finite element method differs significantly from the best approximation with increasing wavenumber and claimed that it is impossible to eliminate the so-called pollution effect. A coupled finite-infinite element method was described, formulated and analyzed for parallel computations by Autrique and Magoulés [2]. Bao et. al. [5] considered the the pollution effect and explored the feasibility of a local spectral method, the discrete singular convolution algorithm for solving the Helmholtz equation with high wavenumbers. Recently, Gitteson et.

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al. [8] proposed discontinous Galerkin finite element methods to capture the oscillatory behavior of the wave solution. It should be pointed out that all numerical methods require a very fine mesh in order to ensure the accuracy of the computed solutions at high wavenumbers.

The mathematical formulation for time harmonic wave propagation in the homogeneous media is given by the Helmholtz equation:

$$\Delta U + k^2 U = 0. \tag{1}$$

where $k = \omega/c$ denotes the wavenumber which is related to the frequency of the wave propagation ω and c is the speed of sound.

Even though tremendous progress has been reported in the areas of computational techniques for partial differential equations, solving a linear Helmholtz equation at high wavenumbers numerically remains as one of the most difficult tasks in scientific computing. At a high wavenumber, the solution of the Helmholtz equation is highly oscillatory. Suppose the mesh size of a numerical discretization is h, it has generally been recognized that to accurately capture the oscillatory behavior, it is necessary to require kh to be small. However, numerical simulation and theoretical study has confirmed that even when kh is fixed, the numerical accuracy deteriorates rapidly as k increases. This is known as the "pollution effect" [4]. The pollution error can only be eliminated completely for one-dimensional equation, and not for two- and three-dimensional problems. Moreover, to ensure an accurate numerical solution, it is essential to enforce the condition $k^2h < 1$. However, this would imply that the number of the discretized equations is proportional to h^{-3} or k^3. This will then lead to an extremely huge system of linear equations. It should be mentioned that the resulting system is highly indefinite for large wavenumbers, and many iterative techniques such as the conjugate gradient and multigrid methods are not capable of solving the indefinite systems.

Developing efficient and accurate numerical solutions for the Helmholtz equation at high wavenumbers is an active research topic. Although it has been reported in many engineering literatures that using 10 to 12 grid points per wavelength is sufficient to produce a reasonable accuracy for many problems, this general rule, however, can not be used when dealing with Helmholtz equation at highwave numbers.

In this paper, we consider a novel finite difference approach which satisfies exactly the interior points of the Helmholtz equation at any wavenumber. Using the same idea, we also derive the finite difference for the radiation boundary conditions. The most important result presented in this work is that the finite difference scheme is constructed so that the solution of the discretized equations satisfies the solution of the Helmhotz equation exactly at the interior grid points as well as the boundary. Since no discretization error is introduced, the numerical solution can be computed for all wavenumbers even if kh and k^2h is not small. Numerical simulations confirm that the new schemes produce exact numerical solution for one-dimensional problem. For a two-dimensional Helmholtz equation, accurate numerical solutions can be achieved even for the case $kh = 1.5$ and $k^2h = 450$. To our knowledge, the exact finite difference scheme has not been reported and demonstrated for solving Helmotz equation especially for applications to high wavenumbers.

The present study is organized as follows. In Section 2, we consider finite difference approximations for the Helmholtz equation. A novel difference approach is presented, so that the resulting difference equations satisfy exactly the original