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A TRANSFER THEORY ANALYSIS OF APPROXIMATE DECONVOLUTION MODELS OF TURBULENCE

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Abstract. This study considers Pao's transfer theory of turbulence for the family of Approximate Deconvolution Models (ADMs). By taking a different representation of the persistent input of energy into the large scales of the turbulent flow, the Pao theory simplifies somewhat. Analysis of the resulting model is given and it is verified that (after the simplification as was known before it) it is consistent with the important statistics of homogeneous isotropic turbulence. The ADMs have an enhanced energy dissipation and a modification to the kinetic energy which affect the truncation of scales by reducing the models microscale from the Kolmogorov microscale. The energy dissipation can be even more enhanced by the time relaxation and the effects of this term are presented as well.

Key words. energy transfer theory, shell model, turbulence, approximate deconvolution.

1. Introduction

Turbulent flows consist of complex, interacting three dimensional eddies of various sizes. In 1941 Kolmogorov gave a remarkable, universal description of the eddies in turbulent flow by combining a judicious mix of physical insight, conjecture, mathematical and dimensional analysis. In his description, the largest eddies are deterministic in nature. Those below a critical size are dominated by viscous forces, and die very quickly due to these forces. This critical length, the Kolmogorov microscale, is $\eta = O(Re^{-3/4})$ in $3d$, so the persistent eddies in a $3d$ flow requires taking $\Delta x = \Delta y = \Delta z = O(Re^{-3/4})$ giving $O(Re^{+9/4})$ mesh points in space per time step. Therefore, direct numerical simulation of turbulent flows (down to the Kolmogorov microscale) is often not computationally economical or even feasible. On the other hand, the largest structures in the flow (containing most of the flow's energy) are responsible for much of the mixing and most of the flow's momentum transport. Thus, various turbulence models are used for simulations seeking to predict a flow's large structures.

One of the mysteries of turbulence is how energy is transferred between scales and how nonlinearity achieves a balance between the input of energy at large scales and its dissipation on exceedingly small scales. In the study of energy transfer among scales, the energy at time t and in scales parameterized by wave-number k, is denoted $E(k,t)$. Energy transfer theories explore this through simplified partial differential equations for $E(k,t)$. Shell models explore the energy transfer among scales by further discretizing the variable k through simplified systems of ordinary differential equations for the energy in a wave-number shell, typically denoted $E_n(t)$ or $u_n(t)$. Transfer theories and shell models have a common aim of understanding a critical feature of turbulent flow and have attracted the attention of many researchers on turbulence so there are a large number of different such models of increasing complexity. Perhaps surprisingly, of these only the simplest Energy Transfer Model of Pao [25] gives unequivocally correct (to the extent that

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the phenomena is understood) predictions of the time averaged statistics and energy spectrum of homogeneous isotropic turbulence. Understanding the mystery of energy transfer through nonlinearity becomes of critical importance in predictions of turbulent flows because one fundamental role of turbulence models is to add $O(1)$ terms which exactly emulate the effects of this not well understood process on scales much larger than the process itself occurs. For example, in 1960 J. Smagorinsky wrote:

> *"In setting up a finite difference grid or a finite wave number space, a turbulent threshold is in effect defined and the question is: How do the equations know how to communicate with the molecular dissipation range? One of course finds empirically that, without any provision for dissipation, the cascade of energy to the higher wave numbers ultimately increases the energy of the smallest wave resolvable by the grid. This energy has no place further to go, and ultimately the calculation departs from nature sufficiently to give intolerable truncation error."* ——— *J. Smagorinsky, 1960*

One promising approach to the simulation of turbulent flows is called *Large Eddy Simulation* or *LES*. Approximate deconvolution models in LES have great promise because they are systematic, have high accuracy and a firm theoretical foundation in some critical respects. The goal of this report is to apply the Pao energy transfer theory to these approximate deconvolution models (ADMs) to gain further insight into their predictions of important turbulent statistics. We derive the energy transfer model associated with ADMs. Interestingly, through a change of variable, the wave-number closure that arises in ADMs becomes exactly the same as the one occurring for the NSE. Thus the Pao closure can be used exactly for the ADM without modification or extra tuning parameters. We thus study the predictions of the Pao transfer theory for ADMs and compare them both theoretically and computationally to those of the NSE. Interestingly, the computational study herein involves wave-number discretization of $E(k,t)$ on wave-number shells (following an equi-partition of energy) and thus results in an apparently new Pao shell model for turbulence.

1.1. The LES Models Considered. In LES the evolution of local, spatial averages over length scales $l \geq \delta$ are sought where δ is user selected. The selection of this averaging radius δ is determined typically by computational resources (δ must be related to the finest computationally feasible mesh), turnaround time needed for the calculation, and estimates of the scales of the persistent eddies needed to be resolved for an accurate simulation. On the face of it, LES seems feasible since the large eddies are believed to be deterministic. The small eddies (accepting Kolmogorov's description) have a universal structure so, in principle, their mean effects on the large eddies should be model-able. The crudest estimate of cost is

$$\Delta x = \Delta y = \Delta z = O(\delta), \tag{1}$$

with thus $O(\delta^{-3})$ storage required in space per time step. On the other hand, it is entirely possible that the computational mesh must be smaller than $O(\delta)$ to predict the $O(\delta)$ structures correctly. It is also entirely possible that, since LES models are themselves inexact and uncertain, solutions to an LES model contain persistent energetic structures smaller than $O(\delta)$. Thus, a good LES model will (i) truncate scales so that *microscale* $= O(\delta)$, consistent with (1), (ii) predict the correct time averaged statistics over scales $l \geq \delta$ (so that computational resolution is free to capture non-universal, non-isotropic, non-fully developed features) and