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QR-BASED METHODS FOR COMPUTING LYAPUNOV EXPONENTS OF STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. Lyapunov exponents (LEs) play a central role in the study of stability properties and asymptotic behavior of dynamical systems. However, explicit formulas for them can be derived for very few systems, therefore numerical methods are required. Such is the case of random dynamical systems described by stochastic differential equations (SDEs), for which there have been reported just a few numerical methods. The first attempts were restricted to linear equations, which have obvious limitations from the applications point of view. A more successful approach deals with nonlinear equation defined over manifolds but is effective for the computation of only the top LE. In this paper, two numerical methods for the efficient computation of all LEs of nonlinear SDEs are introduced. They are, essentially, a generalization to the stochastic case of the well known QR-based methods developed for ordinary differential equations. Specifically, a discrete and a continuous QR method are derived by combining the basic ideas of the deterministic QR methods with the classical rules of the differential calculus for the Stratanovich representation of SDEs. Additionally, bounds for the approximation errors are given and the performance of the methods is illustrated by means of numerical simulations

Key words. Lyapunov Exponents, Stochastic Differential Equations, QR-decomposition, numerical methods.

1. Introduction

Since A.M. Lyapunov introduced the concept of characteristic exponents [27], it has played an important role in the study of the asymptotic behavior of dynamical systems. In particular, the Lyapunov exponents (LEs) have been extensively used for analyzing stability properties of dynamical systems [1]. Some other important contributions like [4] and [5] extend the classical theory of LEs from deterministic dynamical systems to Random Dynamical Systems (RDS). This has allowed the stability analysis of a wide class of physical, mechanical and engineering processes, where the randomness becomes an essential issue for modeling their dynamics [32], [3], [36], [39], [30], [20], [34], [26], [9].

It is well known that, with the exception of some simple cases as those mentioned in [2], explicit formulas for the LEs of Stochastic Differential Equations (SDEs) are rarely known. Alternatively, some different analytic expansions have been obtained for equations driven by particular sources of noise [37]. On the other hand, asymptotic expansions in terms of noise intensity have been obtained for LEs of two-dimensional equations driven by a small noise [6], [33]. Some other asymptotic expansions for LEs have been also obtained in [3] for more general cases of large, small and slow noises. Other types of parametric expansions have been reported for certain particular systems [22], [23], [8]. In principle, the truncation of any of such expansions could be used as a numerical method for computing LEs. However this procedure lacks generality since it is only applicable for certain particular cases of SDEs.

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Indeed, the development of numerical methods for computing LEs of general SDEs is a relevant issue that has not received a systematic attention. In fact, just a few papers have addressed such subject with a relative success [36], [19], [38]. The seminal works in this respect are the numerical method proposed in [36] and [38] for the class of linear stochastic equations. Afterward, this method was extended to the general case of nonlinear SDEs defined either on a compact orientable manifold or on $\mathbb{R}^d$ [19]. It was based on the discretization of the SDE by particular integrators and the corresponding approximation of the LEs for the resulting ergodic Markov chains. From a practical point of view, this method is just effective for the computation of the top LE, but it leads to numerical instabilities for the remaining ones [36].

The aim of this paper is to introduce two alternative methods for the numerical computation of the LEs of nonlinear SDEs defined on $\mathbb{R}^d$. These methods are, essentially, a generalization to the stochastic case of the well known QR-based methods for the computation of the LEs of Ordinary Differential Equations (ODEs) [10], [11], [18], [16], [17]. Specifically, a discrete and a continuous QR method are obtained by combining the basic ideas of the deterministic methods with the classical rules of the differential calculus for the Stratanovich representation of SDEs. In particular, the discrete QR method generalizes the algorithm presented in [19] for the top LE. Moreover, in contrast with that algorithm, the methods introduced here allow the efficient computation of all LEs.

The outline of the paper is as follows. Basic notations as well as essential facts about LEs of SDEs are presented in Section 2. In Section 3 the discrete and continuous QR methods are derived, whereas bounds for the approximation errors are given in Section 4. In section 5, some suggestions for the implementation of the QR methods are presented. Finally, in the last section, the performance of the methods is illustrated by means of simulations.

2. Preliminaries

2.1. Notations. For any matrix $\mathbf{A}$, let us denote by $\mathbf{A}^k$, $\mathbf{A}^{kl}$, $\mathbf{A}^{|k}$ and $\mathbf{A}^{|kk}$ the $k-th$ column vector, the entry (k,l), the matrix of first k columns and the matrix of the first k rows and columns of $\mathbf{A}$, respectively. The Frobenius norm for matrices shall be denoted by $\|\cdot\|$, and the standard scalar product in $\mathbb{R}^d$ by $<\cdot,\cdot>$. For any $k \in \mathbb{Z}^+$ and $0 \leq \delta \leq 1$ let $C_b^{k,\delta}$ be the Banach space of C^k vector fields on $\mathbb{R}^d$ with growth at most linearly and bounded derivatives of order 1 up to k, and whose $k-th$ derivatives are globally δ-Holder continuous. Finally, denote by $\mathcal{C}_P^l(\mathbb{R}^d,\mathbb{R})$ the space of l time continuously differentiable functions $g:\mathbb{R}^d \to \mathbb{R}$ for which g and all its partial derivatives up to order l have polynomial growth.

2.2. Lyapunov Exponents of Nonlinear SDEs. Let $\{\mathcal{F}_t, t \geq 0\}$ be an increasing right continuous family of complete sub σ-algebras of $\mathcal{F}$ and let $(\Omega, \mathcal{F}, \mathcal{P})$ be the canonical Wiener space on $\mathbb{R}^+$ (i.e., $\Omega = \{\omega \in C(\mathbb{R},\mathbb{R}^m) : \omega(0) = 0\}$, $\mathcal{F} = \mathcal{B}(\Omega)$ is the Borel σ-algebra of Ω, and $\mathcal{P}$ is the Wiener measure in $\mathcal{F}$).

Consider the Stratanovich SDE on $\mathbb{R}^d$,

$$d\mathbf{x}_t = \sum_{i=0}^{m} \mathbf{f}_i(\mathbf{x}_t) \circ dw_t^i, \quad \mathbf{x}_{t_o} = \mathbf{x}_0, \quad t \geq t_0 \geq 0, \tag{1}$$

and the variational equation on $\mathbb{R}^d \times \mathbb{R}^d$,