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ASYMPTOTIC EXPANSION AND SUPERCONVERGENCE FOR TRIANGULAR LINEAR FINITE ELEMENT ON A CLASS OF TYPICAL MESH

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Abstract. In this paper, we present a new approach to obtain the asymptotic expansion and superconvergence for the linear element on Union Jack mesh. First, we construct a generalized interpolation function and its discrete harmonic extension by using the energy embedding method and the method of separation of variables. Then, we present elaborate estimates for the generalized interpolation function and the harmonic extension. Finally, the asymptotic expansion, superconvergence and extrapolation are obtained based on those estimates.

Key words. Finite element, Superconvergence, Asymptotic expansion, Extrapolation, Discrete harmonic extension.

1. Introduction

The research of superconvergence of triangular linear finite element can be traced to [2,3] in the late 1970's, but the work was only aimed at uniform meshes in the sense of 3-directional parallel. Later, the superconvergence results of triangular quadratic element were also obtained for uniform meshes in the sense of 3-directional parallel[15].

In the middle of 1990's the local symmetry theory developed by Schatz et.al [12] verified that on the quasi-uniform meshes, the finite element solution has the superconvergence at the mesh symmetry points far from the boundary. Babuska et.al [1] considered four typical quasi-uniform mesh patterns: Regular mesh, Criss-Cross mesh, Chevron mesh and Union Jack mesh. Applying "computer based proof", they found that the mesh symmetry points on the whole domain are derivative superconvergent points for the triangular linear finite element.

In recent years, superconvergence of finite element methods has been a subject of active research due to its strong relevance with a posteriori error estimations for the adaptive finite element method. Lin and Zhang in [10] demonstrated that under the above mentioned four mesh patterns, the mesh symmetry points are "almost" all superconvergent points for linear and high order elements. The authors of [4] and [15] proved that the mesh symmetry points are all superconvergent points for Chevron and Criss-Cross triangular linear finite elements.

In the above four meshes, the topological structure of the Union Jack mesh is comparatively complex. Furthermore, for general mesh with periodic structure, it is difficult to get the superconvergence, extrapolation and asymptotic expansion by the traditional methods, because it can't accurately characterize the influence of the Dirichlet boundary to the finite element asymptotic states. For example, numerical experiment indicates that the symmetry points near the boundary for triangular

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cubic element are not superconvergent points. Combining the numerical results of [1], they show the surprising property of the finite element superconvergence.

In this paper, we consider the Union Jack triangular linear element for the Poisson equation with Dirichlet boundary condition. Using the energy embedding method, first we construct a generalized interpolation function, whose value at the node is only high accuracy approximation to the interpolated function. Note that this interpolated function does not belong to the finite element space, because it doesn't satisfy the homogeneous boundary condition. Then, we construct a discrete harmonic extension corresponding to the interpolation function by the method of separation of variables. The difference between this discrete harmonic extension and the interpolation fuction belong to the finite element spaces. By this means, we can get asymptotic expansion, superconvergence and extrapolation. The key idea is to construct the discrete harmonic extension of the generalized interpolation function.

An outline of this paper is as follows. In section 2, we will introduce the Union Jack triangular linear finite element space and construct a generalized interpolation function with high accuracy approximation. In section 3, we introduce the discrete harmonic function and give the characteristic lemma to construct discrete harmonic function by the method of separation of variables. Then in section 4, we get the main theorem about the asymptotic expansion for the Union Jack triangular linear finite element. Also, we present superconvergence and extrapolation results for the Union Jack triangular linear finite element.

2. Union Jack Linear Element and Preliminary Lemmas

As shown in Fig.1, let $\Omega = (0,1) \times (0,1)$, Ω^h be the Union Jack partition of Ω, $2h$ and N be the mesh size and the partition number, respectively, and $N = \frac{1}{2h}$.

Linear finite element space on the Union Jack mesh is defined by

$$V_h = \{v : v \in C(\overline{\Omega}) \cap H_0^1(\Omega), v|_T \in P_1, \forall T \in \Omega^h\},$$

where P_k is the set of polynomials of degree k.

We consider the following model problem

$$\begin{cases} -\triangle u = f, & (x,y) \in \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

The finite element solution of the above equation $u_h \in V_h$ satisfies

$$A(u_h, v_h) = (f, v_h), \qquad \forall v_h \in V_h,$$

where

$$A(u,v) = \int_\Omega \nabla u \nabla v dxdy,\ (u,v) = \int_\Omega uv dxdy.$$

Denote the node index sets by

$$S = \{(k,l) : k,l = 1,2,...,2N-1\},$$

$$\overline{S} = \{(k,l) : k,l = 0,1,...,2N\},$$

$$S_p = \{(k,l) : (k,l) \in S, \frac{k+l+p}{2} \in Z\},$$

$$\overline{S}_P = \{(k,l) : (k,l) \in \overline{S}, \frac{k+l+p}{2} \in Z\},$$

where $p = 0, 1$, and Z represents the set of integers.