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NUMERICAL COMPUTATION OF THE FIRST EIGENVALUE OF THE p-LAPLACE OPERATOR ON THE UNIT SPHERE

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Abstract. In this paper, we discuss a numerical approximation of the first eigenvalue of the p-Laplace operator on the sphere (S^n, g) of $\mathbb{R}^{n+1}$.

Key Words. First eigenvalue, p-Laplace operator, numerical approximation.

1. Introduction

The p-Laplace operator has been extensively studied in recent years, especially in the context of a bounded domain in $\mathbb{R}^n$ [12, 7, 6, 11, 5, 13, 2, 1]. Recently, there has been an increasing interest in the study of this operator - and in particular of its first eigenvalue - in the more general setting of Riemannian manifolds. The aim of this work is to provide numerical approximation of the first eigenvalue of the p-Laplace operator on the sphere (S^n, g) of $\mathbb{R}^{n+1}$, g being the standard Riemannian metric of the sphere, namely the first positive number λ^* such that the following problem admits a non trivial solution in $W^{1,p}(S^n)$

$$\Delta_p^g u = \lambda^* u|u|^{p-2} \text{ in } S^n, \tag{1.1}$$

where $p > 1$. It is well known that λ^* is the minimizer of the associated energy

$$\lambda^* := \min\{\int_{S^n} |\nabla f|^p, \ f \in W^{1,p}(S^n), \ \|f\|_{L^p} = 1, \ \int_{S^n} |f|^{p-2} f = 0\}. \tag{1.2}$$

That is, λ^* is the best constant such that the following Poincaré type inequality holds for any f such that $\int_{S^n} |f|^{p-2} f = 0$:

$$\int_{S^n} |\nabla f|^p \geq \lambda^* \int_{S^n} |f|^p.$$

By [10, Corollaire 3.1], we know that λ^* is also the first eigenvalue of the p-Laplace operator on a semi-sphere with Dirichlet boundary condition

$$\begin{cases} \Delta_p^g u = \lambda^* u|u|^{p-2} & \text{in } \ S_+^n, \\ u = 0 & \text{on } \ \partial S_+^n = S^{n-1}, \end{cases} \tag{1.3}$$

where S_+^n is the upper semi–sphere.

We know the following

1. $\lambda^* \geq \left[\frac{n-1}{p-1}\right]^{p/2}$ for $p \geq 2$. [10, Theorem3.2]
2. $\lambda^* = n$ in the case where $p = 2$.
3. The first eigenfunction u of (1.3) can be chosen to be nonnegative.
4. u is radial: $u = \varphi(\rho)$ where ρ is the geodesic distance from the north pole S_+^n.
5. u is a non increasing function of $\rho \in [0, \pi/2]$, $\varphi(\pi/2) = 0$ and $\varphi'(0) = 0$.

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Of course, one can set the normalization $\varphi(0) = 1$.

From the expression of the spherical Laplacian in polar coordinates, the constant λ^* appears as the unique positive number such that the following problem admits a solution

$$\begin{cases} \varphi_* \in C^2(0,\pi/2) \\ [-\varphi_*']^{p-2}\left[(p-1)\varphi_*'' + (n-1)\dfrac{\cos\rho}{\sin\rho}\varphi_*'\right] = -\lambda^*\varphi_*^{p-1}, \qquad \rho \in (0,\pi/2) \\ \varphi_* \geq 0,\ \varphi_*(0) = 1,\ \varphi_*'(0) = 0,\ \varphi_*(\pi/2) = 0. \end{cases} \tag{1.4}$$

Behavior of the eigenfunction near $\frac{\pi}{2}$ Let's look to the behavior of the solution of (1.4) near $\frac{\pi}{2}$. First, note that if $p < 2$ then $\varphi_*'(\pi/2) = 0$ implies that $\varphi_*''(\pi/2) = 0$ also. Now, if $p > 2$ then putting $t := \frac{\pi}{2} - \rho$ and writing $\varphi_*(\rho) = \varphi_*(\frac{\pi}{2} - t) = at^\alpha + O(t^\alpha)$, with $\alpha > 1$, we get

$$(a\alpha t^{\alpha-1})^{p-2}\left[(p-1)a\alpha(\alpha-1)t^{\alpha-2} + (n-1)a\alpha t^\alpha\right] = -\lambda^*(\alpha t)^{p-1}.$$

Then necessarily, one has $(\alpha-1)(p-2)+\alpha-2 = p-1$, i.e. $\alpha = \frac{2p-1}{p-1} > 2$. In both cases, $p > 2$ or $p < 2$, we have

$$\varphi_*(\pi/2) = \varphi_*'(\pi/2) = \varphi_*''(\pi/2) = 0. \tag{1.5}$$

2. Some monotony properties

By "first positive eigenvalue" problem it is classically meant: given a manifold $\mathcal{M}$, find a couple (λ,φ), λ the least positive possible such that the problem

$$\begin{cases} \Delta_p\varphi = \lambda\varphi|\varphi|^{p-2} & \text{in} \quad \mathcal{M}, \\ \varphi = 0 & \text{on} \quad \partial\mathcal{M}. \end{cases} \tag{2.1}$$

Aiming to point out some monotony properties, we invert the order: given $\lambda > 0$, find a couple $(\mathcal{M},\varphi)$ such that the associated problem admits a solution.

For our purpose, we limit ourselves to geodesic balls, i.e., $\mathcal{M} = B_g(N,\rho)$, where N is the north pole on the unit sphere and $\rho \in (0,\pi)$. The problem can then be formulated as follows: given $\lambda > 0$, find $(\rho_\lambda,\varphi_\lambda)$ so that φ_λ is the unique solution, up to the multiplication by a constant, of the problem (2.1) on $B_g(N,\rho_\lambda)$. This gives directly the following

Proposition 2.1. *For all $\lambda > 0$ there exists a unique $\rho_\lambda \in (0,\pi)$ such that the problem (2.1) admits a unique solution φ_λ on $B_g(N,\rho_\lambda)$ satisfying $\varphi_\lambda(N) = 1$. Moreover, the mapping $\lambda \longmapsto \rho_\lambda$ is continuous decreasing and* $\lim_{\lambda\to 0}\rho_\lambda = \pi$ *and* $\lim_{\lambda\to\infty}\rho_\lambda = 0$.

3. Approximate problem

Fix $\lambda > 0$, $\rho_\lambda \in (0,\pi)$ and φ_λ solution of the following

$$\begin{cases} \varphi_\lambda \in C^2(0,\rho_\lambda), \\ [-\varphi_\lambda']^{p-2}\left[(p-1)\varphi_\lambda'' + (n-1)\dfrac{\cos\rho}{\sin\rho}\varphi_\lambda'\right] = -\lambda\varphi_\lambda^{p-1}, \qquad \rho \in (0,\rho_\lambda), \\ \varphi_\lambda \geq 0, \quad \varphi_\lambda(0) = 1, \quad \varphi_\lambda'(0) = 0, \quad \varphi_\lambda(\rho_\lambda) = 0. \end{cases} \tag{3.1}$$

In order to study Problem (3.1) we transform it into an initial condition problem. Since we have a problem at zero, using development into fractional Taylor series, one find that $\varphi(\rho) = 1 - a\rho^{2+\alpha} + O(\rho^{2+\alpha})$, for ρ near zero, where

$$\alpha := \frac{2-p}{p-1} \qquad \text{and} \qquad a := \frac{p-1}{p}[\lambda/n]^{\frac{1}{p-1}}.$$