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A HYBRID MORTAR METHOD FOR INCOMPRESSIBLE FLOW

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Abstract. In this paper, we consider the discretization of the Stokes problem on domain partitions with non-matching meshes. We propose a hybrid mortar method, which is motivated by a variational characterization of solutions of the corresponding interface problem. The discretization of the subdomain problems is based on standard inf-sup stable finite element pairs and additional unknowns on the interface. These allow to reduce the coupling between subdomains, which comes from the variational incorporation of interface conditions. The discrete inf-sup stability condition is proven under weak assumptions on the interface mesh, and optimal a-priori error estimates are derived with respect to the energy and L^2-norm. The theoretical results are illustrated with numerical tests.

Key Words. Stokes equations, interface problems, discontinuous Galerkin methods, hybridization, mortar methods, non-matching grids.

1. Introduction

Various applications in computational fluid dynamics involve moving geometries, multiple physical phenomena, or discontinuous material properties. As typical examples, let us mention the flow around spinning propellers, fluid-structure interaction, groundwater contaminant transport, or multiphase flows. For such problems, it may be convenient to use independent discretizations for subdomain problems, which can be non-matching across the interfaces; e.g., in the hydrodynamic simulation of rotating propellers it is common practice to generate independent meshes for the rotor and the stator domain. Continuity of the solution is then obtained by imposing appropriate coupling conditions on the cylindrical interface.

Methods that incorporate interface conditions in a variational framework allow to deal with non-matching meshes more or less automatically. A prominent example are the classical mortar methods [13], which enforce jump conditions across the interface by Lagrange multipliers. Mortar methods are well-studied, cf. e.g. [16, 53], but they have certain peculiarities. For instance, the space of Lagrange multipliers has to be chosen with care in order to retain stability on the discrete level, and the resulting linear systems are of saddlepoint type, and therefore require appropriate solvers.

An alternative variational approach for the discretization of interface problems is offered by Nitsche-type mortaring [11]; see also [31, 50, 34]. Such techniques avoid the use of Lagrange multipliers, and consequently, the resulting linear systems are

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positive definite and can be solved with standard iterative methods like the (preconditioned) conjugate gradient method. A drawback of Nitsche-type mortaring is that a lot of coupling is introduced across the interface, amountingto the large stencils of discontinuous Galerkin methods. This complicates the independent solution of the subdomain problems, and limits the applicability in domain decomposition algorithms [45, 51]. The strong coupling of the subdomain problems can however be relaxed by *hybridization* [25], i.e., by the introduction of additional unknowns at the interface; see also [4, 21] for mixed problems, and also [10, 20] for interface problems and domain decomposition. The hybrid methods yield again positive definite linear systems, and inherit the great flexibility in the choice of ansatz spaces from the Nitsche-type methods, but without introducing their strong coupling.

The aim of this paper is to extend the theoretical framework of hybrid mortar methods [25, 27] to the Stokes system. We derive a variational characterization for the Stokes interface problem, which serves as the starting point for the construction of the hybrid mortar method. The analysis is presented in detail for a two dimensional model problem, and we then discuss how the results can be generalized in order to cover a variety of finite element discretizations in two and three space dimensions. Stability of the discrete problems is obtained under mild conditions on the domain partition; in particular, the meshes on the subdomains can be chosen almost completely independent from each other.

Let us mention further related work: The discretization of Stokes interface problems was investigated in the context of classical mortar methods for instance in [12, 30], and in the framework of discontinuous Galerkin methods in [34, 50, 31]. Hybridization has also been used for the formulation of discontinuous Galerkin methods for Stokes flow [42], and the analysis of the vorticity formulation of Stokes' problem [26]. The approach discussed in this paper however differs in the type of application or discretization. Other aspects of interface problems for Stokes flow, e.g. the use in domain decomposition algorithms, are discussed in [45, 51]; see also [43] for estimates of the inf-sup stability constants independent of jumps in the viscosity.

The plan for our presentation is as follows: In Section 2, we state the Stokes interface problem and derive a variational characterization of solutions to this problem based on a three-field formulation [10, 20]. This characterization is the starting point for the formulation of a hybrid mortar finite element method, and in Sections 3 and 4, we present in detail the stability and error analysis for a specific discretization of a two dimensional model problem. Section 5 then discusses the generalization of the results to three dimensions and more general inf-sup stable finite element spaces. Section 6 finally presents some numerical tests in support of the theoretical results.

2. An interface problem for Stokes flow

2.1. The Stokes problem. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded Lipschitz domain two or three space dimensions. As a model for the flow of an incompressible viscous fluid confined in Ω, we consider the stationary Stokes problem with homogeneous Dirichlet boundary conditions, i.e.,

$$(1) \qquad \left\{ \begin{array}{rcll} -\Delta\, \boldsymbol{u} + \nabla p &=& \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} &=& 0 & \text{in } \Omega, \\ \boldsymbol{u} &=& \boldsymbol{0} & \text{on } \partial\Omega. \end{array} \right.$$