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A STABILIZED NONCONFORMING QUADRILATERAL FINITE ELEMENT METHOD FOR THE GENERALIZED STOKES EQUATIONS

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Abstract. In this paper, we study a local stabilized nonconforming finite element method for the generalized Stokes equations. This nonconforming method is based on two local Gauss integrals, and uses the equal order pairs of mixed finite elements on quadrilaterals. Optimal order error estimates are obtained for velocity and pressure. Numerical experiments performed agree with the theoretical results.

Key words. Generalized Stokes equations, nonconforming quadrilateral finite elements, optimal error estimates, *inf-sup* condition, numerical experiments, stability

1. Introduction

Much attention has recently been attracted to using the equal order finite element pairs (e.g., $P_1 - P_1$–the linear function pair and $Q_1 - Q_1$–the bilinear function pair) for the fluid mechanics equations, particularly for the Stokes and Navier-Stokes equations [1, 10, 11, 12]. While they do not satisfy the *inf-sup* stability condition, these element pairs offer simple and practical uniform data structure and adequate accuracy. Many stabilization techniques have been proposed to stabilize them such as penalty [7, 8], pressure projection [1, 10], and residual [15] stabilization methods. Among these methods, the pressure projection stabilization method is a preferable choice in that it is free of stabilization parameters, does not require any calculation of high-order derivatives or edge-based data structures, and can be implemented at the element level. As formulated in [1, 10, 11, 14], it is based on two local Gauss integrals.

Nonconforming finite elements [4] are popular for the discretization of partial differential equations since they are simple and have small support sets of basis functions. These elements on triangles have been studied in the context of the pressure projection stabilization method [9]. However, due to a technical reason, the nonconforming finite elements on quadrilaterals have not been studied for this stabilization method. In this paper, an argument is introduced to study this class of nonconforming finite elements for the stabilization method of the generalized Stokes equations. As examples, the nonconforming rotated element span$\{1,x,y,x^2-y^2\}$ [3, 13] and the element span$\{1,x,y,((3x^2-5x^4)-(3y^2-5y^4))\}$ proposed by Douglas et al. [6] will be analyzed. After a stability condition is proven for the pressure projection stabilization method, optimal order error estimates are obtained for velocity and pressure. Numerical experiments will be performed to check the theoretical results derived.

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An outline of this paper is given as follows: In the second section, we introduce some basic notation and the generalized Stokes equations. Then, in the third section, the nonconforming quadrilateral finite elements and the local stabilization method are given. In the fourth section, a stability result is shown. Optimal order error estimates are derived in the fifth section. Finally, numerical experiments are presented in the sixth section.

2. Preliminaries

We consider the following generalized Stokes problem:

$$\begin{cases} \sigma u - \nu\Delta u + \nabla p = f & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω represents a polygonal convex domain in $\Re^2$ with a Lipschitz-continuous boundary $\partial\Omega$, $u(x) = (u_1(x), u_2(x))$ the velocity vector, $p(x)$ the pressure, $f(x)$ the prescribed force, $\nu > 0$ the viscosity, and $\sigma \geq 0$ a nonnegative real number. For a time dependent problem, for example, σ can represent a time step.

To introduce a weak formulation of (2.1), set

$$X = (H_0^1(\Omega))^2, \quad Y = (L^2(\Omega))^2,$$
$$M = \left\{ q \in L^2(\Omega) : \int_\Omega q dx = 0 \right\}.$$

Below the standard notation is used for the Sobolev space $W^{m,r}(\Omega)$, with the norm $\|\cdot\|_{m,r}$ and the seminorm $|\cdot|_{m,r}$, $m, r \geq 0$. We will write $H^m(\Omega)$ for $W^{m,2}(\Omega)$ and $\|\cdot\|_m$ for $\|\cdot\|_{m,2}$ when $r = 2$. The spaces $(L^2(\Omega))^m$, $m = 1, 2, 4$, are endowed with the $L^2(\Omega)$-scalar product $(\cdot,\cdot)$ and $L^2(\Omega)$-norm $\|\cdot\|_0$, respectively, as appropriate. Also, the space X is equipped with the scalar product $(\nabla u, \nabla v)$ and the norm $|u|_1$, $u, v \in X$. Because of the norm equivalence between $\|\cdot\|_1$ and $|\cdot|_1$ on X, we sometimes use the same notation for them.

We define the continuous bilinear forms:

$$a(u, v) = \sigma(u, v) + \nu(\nabla u, \nabla v) \qquad \forall\, u,\ v \in\ X,$$
$$d(v, p) = (\nabla \cdot v, p) \qquad \forall v \in X,\ p \in\ M.$$

Now, the variational formulation of problem (2.1) is to find a pair $(u, p) \in X \times M$ such that

$$B((u, p), (v, q)) = (f, v) \qquad \forall (v, q) \in X \times M, \tag{2.2}$$

where

$$B((u, p), (v, q)) = a(u, v) - d(v, p) - d(u, q).$$

The bilinear form $d(\cdot,\cdot)$ satisfies the *inf-sup* condition [4]:

$$\sup_{0 \neq v \in X} \frac{|d(v, q)|}{\|v\|_1} \geq \beta \|q\|_0, \quad q \in M,$$

where β is a positive constant depending only on the domain Ω.