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AN OPTIMAL-ORDER ERROR ESTIMATE FOR A FINITE DIFFERENCE METHOD TO TRANSIENT DEGENERATE ADVECTION-DIFFUSION EQUATIONS

TONGCHAO LU AND JINHONG JIA

Abstract. We prove an optimal-order error estimate in a degenerate-diffusion weighted energy norm for implicit Euler and Crank-Nicolson finite difference methods to two-dimensional time-dependent advection-diffusion equations with degenerate diffusion. In the estimate, the generic constants depend only on certain Sobolev norms of the true solution but not on the lower bound of the diffusion. This estimate, combined with a known stability estimate of the true solution of the governing partial differential equations, yields an optimal-order estimate of the finite difference methods, in which the generic constants depend only on the Sobolev norms of the initial and right-hand side data.

Key Words. convergence analysis, degenerate advection-diffusion equations, finite difference methods, optimal-order error estimates

1. Introduction

In this article we study a classical problem of an optimal-order error estimate for the numerical methods for time-dependent advection-diffusion equations with degenerate diffusion. Time-dependent nonlinear degenerate advection-diffusion equations typically arise in a coupled system of partial differential equations that models immiscible displacement of oil by water in secondary oil recovery processes and the movement of non-aqueous phase liquid in groundwater in subsurface porous media [2, 3, 13, 17, 22]. The time-dependent linear degenerate advection-diffusion equation studied in this paper is a linearized version of the nonlinear problems.

Optimal-order error estimates for Galerkin finite element methods to nondegenerate parabolic or advection-diffusion equations dated back to 1970s via the introduction of Ritz projection [44]. Optimal-order error estimates for Eulerian-Lagrangian finite element or finite difference methods to transient advection-diffusion equations were proved in [10, 11, 26, 30, 39]. Optimal-order error estimates for the coupled systems of advection-diffusion equations with the coupled pressure equation can be found in [8, 15, 27, 36]. The advantage of these estimates is that they are valid for any regular partition. However, because the approximation property of Ritz projection depends on the Peclet number of the problem, so these error estimates also depend on the Peclet number of the problem and could potentially blow up as the lower bound of the diffusion approaches to zero. These estimates do not fully reflect the utility of the methods which were observed computationally [1, 14, 28, 29].

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ε uniform error estimates were sought. In the context of steady-state advection-diffusion equations, optimal-order error estimates were obtained for finite element or difference methods on Shishkin meshes [16, 23]. In the context of Eulerian-Lagrangian methods for transient advection-diffusion equations, ε uniform error estimates in [4, 21, 31, 32, 33, 35, 43, 41, 42]. Recently, a uniformly optimal-order error estimate was proved for a Eulerian-Lagrangian method for time-dependent advection-diffusion equations with degenerate diffusion [38, 41]. In the context of finite element methods to time-dependent advection-diffusion equation, an ε-uniform optimal-order error estimate was proved in [20]. The authors proved a uniformly optimal-order error estimate for a finite element method for degenerate convection-diffusion equations [18].

In this paper we prove an optimal-order error estimate for a space-centered finite difference method with implicit Euler or Crank-Nicolson temporal discretization for time-dependent advection-diffusion equations with degenerate diffusion. In the estimate, the generate constants depend only on certain Sobolev norms of the true solution but not on the lower bound of the diffusion. This estimate, combined with a known stability estimate of the true solution of the governing partial differential equations in [41], yields an optimal-order estimate of the finite element method, in which the generic constants depend only on the Sobolev norms of the initial data and right-hand side data. The rest of this article is organized as follows. In section 2, we formulate the problem and recall preliminary results that are to be used in the paper. In section 3, we prove the optimal-order error estimate for the problem. In section 4, we summarize the results and draw concluding remarks. In section 5, we prove auxiliary lemma that are used in the analysis.

2. Problem Formulation and Preliminaries

2.1. Model problem. We consider a time-dependent convection-diffusion equation in two space-dimensions

$$
\begin{array}{rcll}
u_t + \nabla \cdot \big(\boldsymbol{v}(\boldsymbol{x},t)u - D(\boldsymbol{x},t)\nabla u\big) & = & f(\boldsymbol{x},t), & (\boldsymbol{x},t) \in \Omega \times (0,T], \\
u(\boldsymbol{x},t) & = & 0, & (\boldsymbol{x},t) \in \Gamma \times (0,T], \\
u(\boldsymbol{x},0) & = & u_0(\boldsymbol{x}), & \boldsymbol{x} \in \Omega.
\end{array} \tag{1}
$$

Here $\Omega = (a,b) \times (c,d)$ is a rectangular domain, $\Gamma := \partial\Omega$ is the spatial boundary of Ω. $\boldsymbol{x} = (x,y)$, $\boldsymbol{v} = (V_1(\boldsymbol{x},t), V_2(\boldsymbol{x},t))$ is a velocity field, $f(\boldsymbol{x},t)$ accounts for external sources and sinks, $u_0(\boldsymbol{x})$ is a prescribed initial data, and $u(\boldsymbol{x},t)$ is the unknown solute concentration of a dissolved function. $D(\boldsymbol{x},t)$ is a diffusion coefficient with $0 \le D(\boldsymbol{x},t) \le D_{max} < +\infty$ for any $(\boldsymbol{x},t) \in \Omega \times [0,T]$.

2.2. Preliminaries. Let $W_p^k(\Omega)$ consist of functions whose weak derivatives up to order-k are p-th Lebesgue integrable in Ω, and $H^k(\Omega) := W_2^k(\Omega)$. Let $H_0^1(\Omega) := \{v \in H^1(\Omega) : v(\boldsymbol{x}) = 0, \boldsymbol{x} \in \Gamma\}$. For any Banach space X, we introduce Sobolev