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AN OPTIMAL-ORDER ERROR ESTIMATE FOR AN H¹-GALERKIN MIXED METHOD FOR A PRESSURE EQUATION IN COMPRESSIBLE POROUS MEDIUM FLOW

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Abstract. We present an H^1-Galerkin mixed finite element method for the solution of a nonlinear parabolic pressure equation, which arises in the mathematical models for describing a compressible fluid flow process in subsurface porous media. The method possesses the advantages of mixed finite element methods while avoiding directly inverting the permeability tensor, which is important especially in a low permeability zone. We conducted theoretical analysis to study the existence and uniqueness of the numerical solutions of the scheme and prove an optimal-order error estimate for the method. Numerical experiments are performed to justify the theoretical analysis.

Key Words. H^1-Galerkin mixed finite element method, optimal-order error estimates, numerical examples.

1. Introduction

Mathematical models used to describe compressible fluid flow processes in petroleum reservoir simulation and groundwater contaminant transport lead to a coupled system of time-dependent nonlinear partial differential equations. Let $c(\mathbf{x}, t)$ be the concentration of an invading fluid or a concerned solute or solvent, and let $p(\mathbf{x}, t)$ and $\mathbf{u}(\mathbf{x}, t)$ be the pressure and Darcy velocity of the fluid mixture. The mass conservation for the fluid mixture and for the invading fluid as well as Darcy's law leads to the following system of partial differential equations [1, 2, 3]

$$(1.1)\qquad \begin{cases} (a) & \dfrac{\partial}{\partial t}(\phi\rho) + \nabla\cdot(\rho\mathbf{u}) & = \rho q, & \mathbf{x}\in\Omega,\ t\in J, \\ (b) & \dfrac{\partial}{\partial t}(\phi\rho c) + \nabla\cdot(\rho\mathbf{u}c - \rho\mathbf{D}(\mathbf{x},\mathbf{u})\nabla c) & = \bar{c}\rho q, & \mathbf{x}\in\Omega,\ t\in J, \end{cases}$$

$$(1.2)\qquad \mathbf{u} = -\frac{\mathbf{K}}{\mu}(\nabla p - \rho\mathbf{g}), \quad \mathbf{x}\in\Omega,\ t\in J.$$

Here $\Omega\subset\mathbb{R}^d$ refers to a porous medium reservoir with the boundary $\partial\Omega$, $J = (0, T]$ is the time interval, $\phi(\mathbf{x})$ and $\mathbf{K}(\mathbf{x})$ are the porosity and the permeability tensor of the medium, μ and ρ are the viscosity and the density of the fluid mixture, $\mathbf{g}$ reflects the gravitational effect, $q(\mathbf{x}, t)$ is the source and sink term. $\mathbf{D}(\mathbf{x}, \mathbf{u}) = \phi(\mathbf{x})d_m\ \mathbf{I} + d_t|\mathbf{u}| + (d_l - d_t)(u_i u_j)_{i,j=1}^d/|\mathbf{u}|$ is the diffusion-dispersion tensor, with

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d_m, d_t, and d_l being the molecular diffusion, the transverse and longitudinal dispersivities, respectively, and $\mathbf{I}$ is the identity tensor. $\bar{c}(\mathbf{x}, t)$ is specified at sources and $\bar{c}(\mathbf{x}, t) = c(\mathbf{x}, t)$ at sinks.

In the context of compressible fluid flow process, the first term in (1.1a) does not vanish. The flow equation in (1.1) can be expressed as a nonlinear parabolic equation in terms of the pressure p as follows

$$S_p \frac{\partial p}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = \rho q. \tag{1.3}$$

The variable of primary interest in the mathematical model (1.1)-(1.2) is the concentration c in the transport equation in (1.1), which shows the sweeping efficiency in the enhanced oil recovery in petroleum industry or the extent and location of the contaminant and the effect of remediation in groundwater contaminant transport and remediation. Extensive research has been conducted on the numerical methods and corresponding analysis for the transport equation in (1.1), including (but not limited to) [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17].

Nevertheless, an accurate solution of the concentration c requires an accurate recovery of Darcy velocity $\mathbf{u}$ in the flow equation in (1.1) since the advection and diffusion-dispersion in the transport equation are governed by Darcy velocity. However, the flow properties of the porous media often change abruptly with sharp changes in lithology. These sharp changes are accompanied by large changes in the pressure gradient ∇p which, in a compensatory fashion, yields a smooth Darcy velocity $\mathbf{u}$ [3]. The standard finite difference or finite element methods solve the pressure equation (1.3) for the pressure p directly, which is not necessarily smooth due to the rough coefficients in the equation. The pressure p is differentiated and then multiplied by a possibly rough coefficient $\mathbf{K}/\mu$ to determine Darcy velocity $\mathbf{u}$ via (1.2). Therefore, the resulting Darcy velocity $\mathbf{u}$ is often inaccurate, which then reduces the accuracy of the approximation to the concentration c in the transport equation in (1.1).

The mixed finite element method approximates both the pressure p and Darcy velocity $\mathbf{u}$ from a flow or pressure equation in (1.1) simultaneously, yielding an accurate Darcy velocity $\mathbf{u}$ [3, 26, 27, 28]. This is why the mixed method has been used widely in the numerical simulation of porous medium flow, including both incompressible flow [18, 19, 20, 21, 22, 23, 24] and compressible flow [5, 6, 25]. In the mixed formulation, Darcy's law (1.2) is rewritten as $\mu \mathbf{K}^{-1}\mathbf{u} = \nabla p$ and then combined with the flow equation in (1.1) to form a saddle-point problem. Consequently, the mixed formulation could face numerical difficulties in a low permeability zone due to the inversion $\mathbf{K}^{-1}$.

In this paper we continue our previous work in [41] and develop a fully discrete H^1-Galerkin mixed finite element method which combines the H^1-Galerkin formulation [29, 30] and the expanded mixed finite element method [31]. This would solve for the pressure p, its gradient $\boldsymbol{\sigma} = \nabla p$ and Darcy velocity $\mathbf{u} = (\mathbf{K}/\mu)\nabla p$ directly, and thus avoids invert $\mathbf{K}$ explicitly. Furthermore, this formulation permits the use of standard continuous and piecewise (linear or higher-order) polynomials in contrast to continuously differentiable piecewise polynomials required by standard H^1-Galerkin method [29, 30], and is free of LBB condition as required by the mixed finite element method. An optimal error estimate for fully discrete approximation was proved under milder regularity assumptions and the CFL condition. Numerical tests are performed to confirm the theoretical analysis. There have been works in the literature on the development and analysis H^1-Galerkin mixed finite element