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ON THE INACCURACIES OF SOME FINITE VOLUME DISCRETIZATIONS OF THE LINEARIZED SHALLOW WATER PROBLEM

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Abstract. In this article we are interested in the study of the errors introduced by different finite volume discretizations in the study of the wave frequencies. This study is made in the context of hyperbolic systems arising in meteorology and oceanography. We show the existence of significant errors in the dispersion relation, only the long waves being computed accurately; this conclusion is similar to the finite differences case described in the article of Grotjahn and O'Brien quoted below. For the case of inertia-gravity waves, we study three often used schemes which are based on the upwind, centered or Lax-Wendroff fluxes. Moreover, the Total Variation Diminishing method (TVD) made from these fluxes (which usually provides an efficient way to eliminate spurious numerical oscillations) will give the same errors in the dispersion relation.

Key Words. finite volume discretizations, spurious caustics, numerical error, dispersion relation, hyperbolic systems

1. Introduction

The present work examines the errors introduced in low order finite volume discretizations velocity in second order schemes using methods which are typical of finite volume methods, e.g. the Lax-Wendroff method and upwinding. We observe that in addition to the well-known problems in phase propagation and group velocity in waves near the grid scale, the differential errors in the group velocity as a function of wave vector gives rise to the spurious formation of caustics in the inertia gravity part of the spectrum. Serious physical ramifications of spurious caustics in the divergent modes could thus result through the interaction of such local focusing and moist processes in models used for weather and climate prediction.

These results were to some extent foreshadowed in the in the context of finite difference discretizations: in Grotjahn and O'Brien (1976) the authors showed that the finite difference methods are causing significant angular and magnitude errors for the group velocity for different ocean models and only the long waves are represented with reasonable accuracy; Vichnevetsky (1987) studied the error introduced by the time and space discretizations, as well as by the boundaries; Blayo (2000) studied the error introduced by finite-difference schemes of higher orders, in the case of inertia-gravity waves; David, et. al. (2006) examined the linear dispersive mechanism for error focusing and the existence of spurious caustics for the case of discretized propagation-type equations. The motivation for considering the study of finite volume methods is that finite volume methods differ conceptually from finite

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differences in their emphasis on estimating grid cell boundary fluxes as opposed to estimating derivatives for the finite difference techniques.

This study has been limited to low order spatial and temporal discretizations for finite volume methods. A more detailed analysis of finite volume discretizations using higher order numerics is the topic of ongoing research. The study of finite volume methods for hyperbolic problems is a subject of great interest and we mention in particular the books and articles by Levesque (2002), Toro (2006) and Toro (2001) (see also Toro and Clarke (1998)). We also cite here the work of Morton (2001/02) where first, second and third order accurate algorithms are developed for non-uniform one-dimensional grids, as well as for unstructured triangular meshes.

Of high interest in the recent years was the use of high-order finite volume methods for hyperbolic problems and we mention some of the recent works: Dumbser et.al. (2008), Martí-Müller (1996), Noelle et. al. (2007), Popov and Ustyugov (2007). It is already known that important group velocity errors exist in higher order schemes (see David and Sagaut (2008)) which have consequences for the prediction and simulation of coherent structures in the atmosphere related to solitons. The existence of spurious caustics in the shallow water equations for higher order FV discretizations, like those discussed here for second order schemes, has not yet been shown and is the subject of current research.

Of primary interest in this study will be the linearized Shallow Water problem with space periodical boundary conditions:

$$\frac{\partial u}{\partial t} + \bar{u}_0 \frac{\partial u}{\partial x} + \bar{v}_0 \frac{\partial u}{\partial y} + g \frac{\partial \Phi}{\partial x} - fv = 0, \tag{1.1}$$

$$\frac{\partial v}{\partial t} + \bar{u}_0 \frac{\partial v}{\partial x} + \bar{v}_0 \frac{\partial v}{\partial y} + g \frac{\partial \Phi}{\partial y} + fu = 0, \tag{1.2}$$

$$\frac{\partial \Phi}{\partial t} + \bar{u}_0 \frac{\partial \Phi}{\partial x} + \bar{v}_0 \frac{\partial \Phi}{\partial y} + \bar{\Phi}_0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0, \tag{1.3}$$

which can be rewritten as a system:

$$\frac{\partial q}{\partial t} + \mathbf{A} \frac{\partial q}{\partial x} + \mathbf{B} \frac{\partial q}{\partial y} + \mathbf{C} q = 0, \tag{1.4}$$

with $q(x, y, t) = (u(x, y, t), v(x, y, t), \Phi(x, y, t))$, g the gravity and the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ defined by:

$$\mathbf{A} = \begin{pmatrix} \bar{u}_0 & 0 & g \\ 0 & \bar{u}_0 & 0 \\ \bar{\Phi}_0 & 0 & \bar{u}_0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \bar{v}_0 & 0 & 0 \\ 0 & \bar{v}_0 & g \\ 0 & \bar{\Phi}_0 & \bar{v}_0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & -f & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Of special interest below will be the case where $\bar{u}_0 = \bar{v}_0 = 0$ and f constant describing the inertia gravity waves. The properties of inertia-gravity waves are important, for example, in the process of geostrophic adjustment in which the atmosphere responds to changes in surface forcing with a continuous spectrum of high-frequency waves. In the numerical primitive equation models, these waves are generated during initialization and after convective events.

The continuous case. To find the dispersion relation in the continuous case, we replace q in equation (1.4) by the wave $q(x, y, t) = (\bar{u}, \bar{v}, \bar{\Phi}) e^{\mathbf{i}(kx+ly-\omega t)}$, and we