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A ROBIN-ROBIN NON-OVERLAPPING DOMAIN DECOMPOSITION METHOD FOR AN ELLIPTIC BOUNDARY CONTROL PROBLEM

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Abstract. A Robin-Robin non-overlapping domain decomposition method for an optimal boundary control problem associated with an elliptic boundary value problem is presented. The existence of the whole domain and subdomain optimal solutions is proven. The convergence of the subdomain optimal solutions to the whole domain optimal solution is shown. The optimality system is derived and a gradient-type method is defined for finding the optimal solution. A theoretic convergence result for the gradient method is established. The finite element version of the Robin-Robin non-overlapping domain decomposition method is analyzed and some numerical results by the method on both serial and parallel computers (using MPI) are presented.

Key Words. Nonoverlapping domain decomposition method, elliptic boundary control problem, elliptic boundary value problem, MPI.

1. Introduction

Domain decomposition methods have the subject of extensive study in the last few decades; see, e.g., www.ddm.org. An important class of non-overlapping domain decomposition method is the Robin-Robin type methods based on successive exchanges of interface Robin data [10, 11, 7]. In this paper we design and analyze a Robin-Robin non-overlapping domain decomposition method for solving an optimal boundary control problem constrained by the second order elliptic partial differential equation (PDE). In addition, we develop both serial and parallel (MPI) codes to give some numerical results.

The content of this paper is as follows. In Section 2.1, we introduce the whole-domain and subdomain optimal boundary control problems. In Section 2.2, we prove the existence of the whole domain optimal solution and the subdomain optimal solution. In Section 2.3, we show that the subdomain optimal solution converges weakly to the whole domain optimal solution. In Section 2.4, we use the method of Lagrange multiplier to derive the optimality system of equations. In Section 2.5, we define a gradient method for our optimal boundary control problem on the subdomain and prove the theoretic convergence of the method. In Section 3, we analyzed the finite element version of the Robin-Robin non-overlapping domain decomposition method in the same way as we did the continuous version. Finally, in Section 4, we use both serial and parallel computers to present numerical results. In the parallel computing, we use MPI, the Message Passing Interface, for the communication between computer processors.

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2. Optimal boundary control problems

In this chapter, we solve an optimal control problem constrained by the general second order elliptic PDE under the Neumann boundary condition using the Robin type non-overlapping domain decomposition method (DDM).

2.1. The model problem. We consider the general second order elliptic PDE under the Neumann boundary condition:

$$-\mathrm{div}\,[A(\mathbf{x})\nabla u] + \mathbf{b}(\mathbf{x})\cdot\nabla u + c(\mathbf{x})u = f \quad \text{in } \Omega, \qquad [A(\mathbf{x})\nabla u]\cdot\mathbf{n} = p \quad \text{on } \Gamma, \tag{2.1}$$

where Ω is an open, bounded subset of $\mathbb{R}^2$ with boundary Γ, $u : \bar{\Omega} \to \mathbb{R}$ is the unknown, A is a symmetric-matrix-valued $L^\infty(\Omega)$ function that is uniformly positive definite, $\mathbf{b}$ is a vector-valued $L^\infty(\Omega)$ function, c is a real-valued $L^\infty(\Omega)$ function, $f \in L^2(\Omega)$, $\mathbf{n}$ is the outward normal to Ω, and $p \in L^2(\Gamma)$ is a flexible boundary input data called a boundary control.

Here, we optimize the following cost functional subject to (2.1):

$$\mathcal{J}_\beta(u,p) = \frac{1}{2}\int_\Omega |u-U|^2 d\Omega + \frac{\beta}{2}\int_\Gamma p^2 d\Gamma, \tag{2.2}$$

where U is a given target solution and β is a positive constant.

In this paper, in order to minimize (2.2) using DDM, we partition the whole domain Ω into two subdomains Ω_1 and Ω_2. Then we denote a new boundary by Γ_0, separate the original boundary into $\Gamma_1 = \partial\Omega_1\backslash\Gamma_0$ and $\Gamma_2 = \partial\Omega_2\backslash\Gamma_0$. (see Figure 2.1)

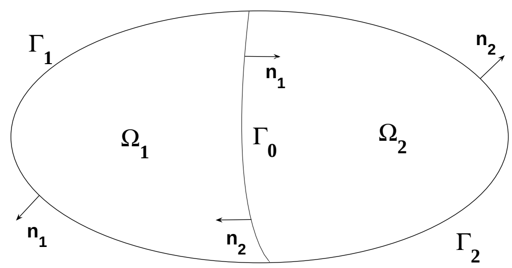

FIGURE 2.1. Two subdomains

In Figure 2.1, $\mathbf{n}_i$ is the unit outward normal to $\partial\Omega_i$.

Before we solve this problem, we introduce notation:

$$[u,v]_{\mathcal{D}} = \int_{\mathcal{D}} uv\; d\mathcal{D} \quad \forall u,v \in L^2(\mathcal{D}) \quad \text{and}$$

$$a[u,v] = \int_\Omega [A(\mathbf{x})\nabla u\cdot\nabla v + (\mathbf{b}(\mathbf{x})\cdot\nabla u)v + c(\mathbf{x})uv]\; d\Omega \quad \forall u,v \in H^1(\Omega),$$

where $H^1(\Omega)$ is the standard Sobolev space (see [1]).

Under the notation, we have the weak formulation of (2.1): seek $u \in H^1(\Omega)$ such that

$$a[u,v] = [f,v]_\Omega + [p,v]_\Gamma \quad \forall v \in H^1(\Omega). \tag{2.3}$$

We assume here that, throughout the paper, our bilinear forms are coersive; e.g., in (2.3), there is a constant $C > 0$ such that

$$a[u,u] \geq C\|u\|^2_{1,\Omega} \quad \forall u \in H^1(\Omega) \tag{2.4}$$

to ensure the existence of the solution of our PDE.