INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 8, Number 1, Pages 70–85

ON ERROR ESTIMATES OF THE PRESSURE-CORRECTION PROJECTION METHODS FOR THE TIME-DEPENDENT NAVIER-STOKES EQUATIONS

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Abstract. In this paper, we present a new pressure-correction projection scheme for solving the time-dependent Navier-Stokes equations, which is based on the Crank-Nicolson extrapolation method in the time discretization. Error estimates for the velocity and the pressure of semidiscretized scheme are derived by interpreting the projection scheme as second-order time discretization of a perturbed system which approximates the incompressible Navier-Stokes equations.

Key Words. Navier-Stokes equations, projection method, pressure-correction, Crank-Nicolson extrapolation scheme, error estimates.

1. Introduction

Let Ω be a bounded domain in R^2 assumed to have a sufficiently smooth boundary $\partial\Omega$. Now we consider the time-dependent Navier-Stokes problem

$$\begin{cases} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \ (x,t)\in\Omega\times(0,T], \\ \text{div } u = 0, \ (x,t)\in\Omega\times(0,T], \\ u(x,0) = u_0(x), \ x\in\Omega, \end{cases} \tag{1}$$

where $u = u(x,t)$ represents the velocity vector of a viscous incompressible fluid, $p = p(x,t)$ the pressure, $f = f(x,t)$ the prescribed body force. The problem (1) should be completed with an appropriate boundary condition for the velocity u. For the sake of convenience, we consider the homogeneous Dirichlet boundary condition, i.e. $u|_{\partial\Omega} = 0, \forall\ t\in(0,T]$.

It is well known that the numerical solution of problem (1) involves several major difficulties, and the crucial difficult is that the unknowns u and p are coupled through the incompressibility condition div $u = 0$. Generally, in order to overcome this difficulty, people often relax the incompressibility constraint in an appropriate way, resulting in a class of pseudo-compressibility methods, among which are the penalty method, the artificial compressibility method, the pressure stabilization method and the projection method, see for instance [1, 2, 4, 7, 12, 17, 20, 22]. The projection method is perhaps the most efficient and the easiest to implement for solving the time-dependent Navier-Stokes equations.

The original projection method was introduced by Chorin [4] and Temam [26] respectively in the late 60s. The original method is simple, but is not satisfactory

Received by the editors September 22, 2009 and, in revised form May 14, 2010.

2000 *Mathematics Subject Classification.* 35S05, 35A35, 65M60, 65N22.

This work is in parts supported by the NSF of China (No.10971166, No.10901131) and the National Basic Research Program (No.2005CB321703).

since its convergence rate is irreducibly limited to $O(\delta t)$. In order to solve these problems, many literatures are put into the construction, analysis and implementation of projection-type schemes, (see for instance [6, 8, 10, 11, 16, 18, 20, 25, 27]). An important class of projection methods is the so-called pressure correction methods introduced in [3, 8, 28]. These schemes consist of two substeps per time step: the pressure is treated explicitly in the first substep and corrected in the second substep by projecting the intermediate velocity onto the space of divergence-free fields. These schemes are widely used in practice and have been rigorously analyzed in [5, 9, 24].

The goal of this paper is to present a rigorous error analysis for the standard incremental pressure-correction scheme, which is based on the Crank-Nicolson extrapolation method in the time discretization. We prove the stability and second order convergence in the L^2-norm of the velocity, and first order convergence in the L^∞-norm of the pressure. Our results are consistent with the reference [23], it appear to be the best possible under the general context considered in this paper.

The remainder of this paper is organized as follows. In Section 2, we introduce some notations and recall important results which are used repeatedly in the core of this paper. In Section 3, we give the new pressure-correction projection scheme for solving the incompressible time-dependent Navier-Stokes equations, and we prove the stability of the scheme. In Section 4, we derive some additional a priori estimates for (u^n, p^n) and perform some error analysis.

2. Preliminaries

For the mathematical setting of problem (1), we introduce the following Hilbert spaces:

$$X = H_0^1(\Omega)^2,\ Y = L^2(\Omega)^2,\ W = L_0^2(\Omega) = \left\{ q \in L^2(\Omega); \int_\Omega q(x)dx = 0 \right\}.$$

The space Y is equipped with the usual L^2-scalar product $(\cdot,\cdot)$ and L^2-norm $\|\cdot\|_0$. Denote by $\|\cdot\|_r$ the norm on Sobolev spaces $H^r(\Omega)^2$, where $r = 1, 2$. We recall that if Ω is bounded in some direction then the Poincaré inequality holds:

$$\|v\|_0 \le c(\Omega)\|\nabla v\|_0, \quad \forall v \in X.$$

The quotient space $H^1(\Omega)/R$ is defined as follows: the element of the quotient space is equivalence classes. That is $\forall v \in H^1(\Omega)$, the equivalence class of v is often denoted

$$\hat{v} = \{u | u \in H^1(\Omega), u - v \in R\}.$$

Next, let the closed subset V of X be given by

$$V = \{v \in X;\ \text{div}\ v = 0 \text{ in } \Omega\},$$

and we denote by H the closed subset of Y, one can show that

$$H = \{v \in Y;\ \text{div}\ v = 0 \text{ in } \Omega \text{ and } v \cdot \vec{n}|_{\partial\Omega} = 0\}.$$

We refer reader to [7, 13-15] for details on these spaces. And P_H is the orthogonal projector in Y onto H, i.e.

$$(u - P_H u, v) = 0, \quad \forall u \in Y,\ v \in H.$$

The following inequalities (cf. [27])

$$\|P_H v\|_i \le c(\Omega)\|v\|_i, \quad \forall v \in H^1(\Omega)^2,\ i = 0, 1 \tag{2}$$