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SIMULATING THE AXISYMMETRIC INTERFACIAL FLOWS WITH INSOLUBLE SURFACTANT BY IMMERSED BOUNDARY METHOD

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Abstract. In this paper, we extend our previous work on the two-dimensional immersed boundary method for interfacial flows with insoluble surfactant to the case of three-dimensional axisymmetric interfacial flows. Although the key components of the scheme are similar in spirit to the two-dimensional case, there are two differences introduced in the present work. Firstly, the governing equations are written in an immersed boundary formulation using the axisymmetric cylindrical coordinates. Secondly, we introduce an artificial tangential velocity to the Lagrangian markers so that the uniform distribution of markers along the interface can be achieved and a modified surfactant concentration equation is derived as well. The numerical scheme still preserves the total mass of surfactant along the interface. Numerical convergence of the present scheme has been checked, and several tests for a drop in extensional flows have been investigated in detail.

Key Words. Immersed boundary method; Axisymmetric interfacial flow; Navier-Stokes equations; Surfactant

1. Introduction

Surfactant are surface active agents that adhere to the fluid interface and affect the interface surface tension. Surfactant play an important role in many applications in the industries of food, cosmetics, oil, etc. For instance, the daily extraction of ore rely on the subtle effects introduced by the presence of surfactant [6]. In a liquid-liquid system, surfactant allow small droplets to be formed and used as an emulsion. Surfactant also play an important role in water purification and other applications where micro-sized bubbles are generated by lowering the surface tension of the liquid-gas interface. In microsystems with the presence of interfaces, it is extremely important to consider the effect of surfactant since in such cases the capillary effect dominates the inertia of the fluids [21]. In bio-mechanical flows, for example, some insects drift on the water by injecting a chemical excretion at the rear to reduce the surface tension behind their bodies such that the insects are pulled forward.

In [11], we develop an immersed boundary method for two-dimensional interfacial flows with insoluble surfactant. In this paper, we extend our previous work to the three-dimensional axisymmetric case where the governing equations are written in immersed boundary formulation [16]. Other related works on the simulations for

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axisymmetric interfacial flows with insoluble surfactant using different methodology including the boundary integral method, volume-of-fluid method, front-tracking method, and the arbitrary Lagrangian Eulerian method can be found in [20, 18, 23, 9, 12, 7, 5], just to name a few. The simulations for axisymmetric interfacial flows with soluble surfactant using the front-tracking method can refer to [25, 15, 24].

For simplicity, we consider an axisymmetric drop immersed in a viscous incompressible fluid. We also assume that the fluids inside and outside of the drop are the same, and they are governed by the incompressible Navier-Stokes equations. Under the axisymmetric assumption, one can describe the drop interface Σ with a parametric form $\mathbf{X}(s,t) = (R(s,t), Z(s,t)), 0 \le s \le 2\pi$, where s is the parameter of the initial configuration of the interface. Note that, s is not necessarily the arc-length parameter. The unit tangent vector $\boldsymbol{\tau}$ and the outward normal vector $\mathbf{n}$ on the interface can be defined as

$$\boldsymbol{\tau}(s,t) = \frac{\boldsymbol{X}_s}{|\boldsymbol{X}_s|} = \frac{(R_s, Z_s)}{\sqrt{R_s^2 + Z_s^2}}, \qquad \mathbf{n} = \frac{(Z_s, -R_s)}{\sqrt{R_s^2 + Z_s^2}}, \tag{1.1}$$

respectively, where the subscript denotes the partial derivative with respect to s.

Under the assumption of axis-symmetry, the 3D Navier-Stokes equations can be simply written in a 2D manner using the axisymmetric cylindrical coordinates $\mathbf{x} = (r, z)$. Throughout this paper, we denote $\boldsymbol{u} = (u, w)$ as the velocity defined on a 2D meridian domain $\Omega = \{(r,z) | 0 < r \le a, c \le z \le d\}$, where u and w are the radial (r coordinate) and axial (z coordinate) velocity components. We also denote $\boldsymbol{U} = (U, W)$ as the corresponding velocity component on the interface. The non-dimensional Navier-Stokes equations are

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial r} + w\,\frac{\partial u}{\partial z} + \frac{\partial p}{\partial r} = \frac{1}{Re}(\Delta u - \frac{u}{r^2}) + \frac{1}{ReCa}\,f_r \tag{1.2}$$

$$\frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial r} + w\,\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} = \frac{1}{Re}\Delta w + \frac{1}{ReCa}\,f_z \tag{1.3}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0 \tag{1.4}$$

where the Laplacian operator is defined as

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}. \tag{1.5}$$

For later convenience, we introduce the gradient and divergence operators as

$$\nabla = \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial z}\right) \qquad \tilde{\nabla}\cdot = \left(\frac{1}{r}\frac{\partial}{\partial r}r, \frac{\partial}{\partial z}\right)\cdot\,. \tag{1.6}$$

Thus, the Laplace operator can be written as $\Delta = \tilde{\nabla}\cdot\nabla$.

The fluid-interface interaction equations can be written as

$$\boldsymbol{f} = (f_r, f_z) = \int_0^{2\pi} \mathbf{F}(s,t)\,\delta^2(\mathbf{x} - \mathbf{X}(s,t))\,ds, \tag{1.7}$$

$$\mathbf{F}\,(s,t) = \frac{\partial}{\partial s}\,(\sigma\,\boldsymbol{\tau}) - \frac{Z_s}{R}\,\sigma\,\mathbf{n} \tag{1.8}$$

$$\frac{\partial \mathbf{X}(s,t)}{\partial t} = \boldsymbol{U}(s,t) = \int_\Omega \boldsymbol{u}(\mathbf{x},t)\,\delta^2(\mathbf{x} - \mathbf{X}(s,t))\,d\mathbf{x}. \tag{1.9}$$

One can immediately see that interfacial force $\mathbf{F}(s,t)$ is slightly different from the two-dimensional counterpart since an extra term is added. A detailed derivation for Eq. (1.8) is given in the Appendix. Notice that, the above immersed boundary formulation is also used in [25].