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A POSTERIORI ERROR ESTIMATION FOR A DEFECT-CORRECTION METHOD APPLIED TO CONVECTION-DIFFUSION PROBLEMS

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Abstract. We consider a two-point boundary-value problem for a singularly perturbed convection-diffusion problem. The problem is solved by using a defect-correction method based on a first-order upwind difference scheme and a second-order (unstabilized) central difference scheme.
A robust *a posteriori* error estimate in the maximum norm is derived. It provides computable and guaranteed upper bounds for the discretization error. Numerical examples are given that illustrate the theoretical findings and verify the efficiency of the error estimator on *a priori* adapted meshes and in an adaptive mesh movement algorithm.

Key Words. convection-diffusion problems, finite difference schemes, defect correction, a posteriori error estimation, singular perturbation

1. Introduction

Defect correction methods (DCM) have been advocated for the numerical solution of ordinary and partial differential equations since the early 1970s and 80s [27, 5]. The idea of DCMs is to combine the good stability properties of a low order upwinded discretization with the higher order accuracy of unstabilized discretizations. They have been successfully applied in computational fluid dynamics, for example to combustion problems [3] or when solving the Navier-Stokes equations [14, 19].

Hemker [12, 13] proposed the use of DCM for the numerical treatment of convection-diffusion and other singularly perturbed boundary-value problems. Most of the papers found in the literature deal with DCM on (quasi)uniform meshes. Only recently adaptivity and layer-adapted meshes have been used in combination with DCM, see [9, 10, 15, 22]. Of particular interest are parameter-uniform methods, i.e., methods that perform equally well no matter how small the perturbation parameter.

Let us consider the convection-diffusion problem

$$\mathcal{L}u := -\varepsilon u'' - (bu)' + cu = f \quad \text{in} \quad (0,1), \quad u(0) = \gamma_0, \; u(1) = \gamma_1, \tag{1}$$

where ε is a small positive parameter and $b \geq \beta > 0$ on $[0,1]$. It provides an excellent paradigm for numerical techniques in computational fluid dynamics for

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the treatment of problems with boundary layers, i. e., regions where the solution and its derivates change rapidly [26].

In the present paper we shall investigate a DCM for (1) based on finite difference discretizations. Ervin and Layton [8] proved that this method is uniformly convergent of second in the maximum norm order outside layers. However, the crucial point in singularly perturbed problems is the resolution of layers. This can for example be achieved by the use of layer-adapted meshes, i.e., meshes that are significantly refined inside the layer regions. The resulting non-uniformity of the mesh results in difficulties both in the appropriate construction of the DCM and its analysis which must be overcome.

In [10] a DCM on a particular class of layer-adapted meshes, so called Shishkin-type meshes, is considered. The authors conduct an *a priori* error analysis and establish uniform nodal convergence of essentially second order in all mesh points. A theory for arbitrary meshes has been derived in [22, 23].

Let us describe the DCM from [10, 22, 23]. Given a mesh $\omega_N : 0 = x_0 < x_1 < \cdots < x_N = 1$ with mesh sizes $h_i := x_i - x_{i-1}$ and $\hbar_i = (h_i + h_{i+1})/2$ define the difference operators

$$v_{x,i} := \frac{v_{i+1} - v_i}{h_{i+1}}, \quad v_{\bar{x},i} := \frac{v_i - v_{i-1}}{h_i}, \quad v_{\hat{x},i} := \frac{v_{i+1} - v_i}{\hbar_i} \text{ and } v_{\mathring{x},i} := \frac{v_{i+1} - v_{i-1}}{2\hbar_i}.$$

Then the central difference approximation on ω_N for (1) is

$$\left[L^c \bar{u}^N\right]_i := -\varepsilon \bar{u}^N_{\bar{x}\hat{x},i+1} - (b\bar{u}^N)_{\mathring{x},i} + c_i \bar{u}^N_i = f_i.$$

It is combined with the upwind scheme

$$\left[L^u \hat{u}^N\right]_i := -\varepsilon \hat{u}^N_{\bar{x}x,i+1} - (b\hat{u}^N)_{x,i} + c_i \hat{u}^N_i = f_i,$$

where for any function $g \in C[0,1]$ we have set $g_i := g(x_i)$.

With this notation the DCM is as follows:

1. Compute an initial first-order approximation $\hat{u}^N$ using simple upwinding:

$$\left[L^u \hat{u}^N\right]_i = f_i \text{ for } i = 1, \ldots, N-1, \quad \hat{u}^N_0 = \gamma_0, \quad \hat{u}^N_N = \gamma_1. \tag{2a}$$

2. Estimate the defect τ in the differential equation by means of the central difference scheme:

$$\tau_i = \left[L^c \hat{u}^N\right]_i - f_i \text{ for } i = 1, \ldots, N-1. \tag{2b}$$

3. Compute the defect correction Δ by solving

$$[L^u \Delta]_i = \kappa_i \tau_i, \quad \kappa_i := \frac{\hbar_i}{h_{i+1}} \text{ for } i = 1, \ldots, N-1, \quad \Delta_0 = \Delta_N = 0. \tag{2c}$$

4. Then the final computed solution is

$$u^N_i = \hat{u}^N_i - \Delta_i \text{ for } i = 0, \ldots, N. \tag{2d}$$

Remark 1. *At a first glance both the upwind discretization and the particular weighting of the residual in* (2c) *appear a bit non-standard. No justification for these choices are provided by* [10, 22, 23]. *An argument that suggests this particular choice is presented in Sect. 4. Furthermore, our weighting becomes the standard* $\kappa_i = 1$ *on uniform meshes; however, when used on non-uniform meshes,* $\kappa_i = 1$ *might reduce the order of convergence (see numerical results in Sect. 5.3).*

While the *a priori* results [10, 22, 23] establish the asymptotic behaviour of the error as the mesh is refined, it cannot give guaranteed upper bounds for the error on a particular mesh. The constant in the error bound, though independent of the perturbation parameter ε, depends on the exact solution u which in turn is unknown.