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ERROR ESTIMATES OF MORLEY TRIANGULAR ELEMENT SATISFYING THE MAXIMAL ANGLE CONDITION

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Abstract. In this paper, we establish the convergence of a nonconforming triangular Morley element for the plate bending problem on degenerate meshes. An explicit bound for the interpolation error is derived for arbitrary triangular meshes without any assumptions. The optimal convergence rates of the moment error and rotation error are derived for triangular meshes satisfying the maximal angle condition. Our results can also be extended to the three dimensional Morley element presented recently in [41]. Finally, some numerical results are reported that confirm our theoretical results.

Key Words. Morley element, plate elements, plate bending problems, maximal angle condition

1. Introduction

It is well known that the regularity assumption on the meshes [13, 16], i.e., bounded ratio between outer and inner diameters, leads to the convergence of standard finite element methods. However the above conventional mesh condition is a severe restriction for some particular problems of recent interests. For instance, for problems for which the solution may have anisotropic behavior in some parts of the domain, that is to say, the solution varies significantly only in certain directions. Such problems are frequently encountered in singularly perturbed convection-diffusion-reaction equations where boundary or interior layers appear or problems set on domains with edges where edge singularities may occur. In such cases, regular meshes are inappropriate or may even fail to give satisfactory results, hence the use of degenerate (or anisotropic) meshes is recommended.

The early mathematical consideration of anisotropic elements goes back to the seventies [11, 21]. Since the end of the eighties, anisotropic elements have been extensively studied. In particular, we refer to Apel et al [4-9], Chen et al [15, 27-29], Durán et al [1-3, 17, 18], Formaggia et al [19, 20], Křížek [22, 23], Kunert [24, 25], Shenk [36], Ženisek [43, 44] and references therein. As applications of anisotropic finite elements, let us quote for example, the investigation of Laplace type problems in domains with edges [5, 7, 8], layers in some singularly perturbed problems [6, 18, 25], anisotropic phenomena in the solution of Stokes and Navier-Stokes problems [9], and anisotropic a posteriori error estimates [20, 24, 25]. From these papers, it is now well known that the regularity assumption on the meshes can be considerably weakened. However, all these references are mainly restricted

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to second order problems. For fourth order problems, the plate bending problem for example, only some rectangular elements have been considered, see [15, 33, 29]. But as far as we know, up to now, there are no results for general anisotropic triangular plate elements.

Triangular plate elements, especially nonconforming ones are very popular. Such elements have more advantages over their rectangular counterparts since they can be better adapted to complex boundaries. The main goal of this paper is to provide error estimates of the well-known nonconforming Morley triangular element under a weak angle condition.

The Morley element is particularly attractive for fourth order problems, because of its simple structure and since it has low degrees of freedom. However, since the continuity of Morley element is very weak (non-C^0 element), even for regular meshes, error analysis is not easy, see [30, 26, 34, 10, 37]. In this paper, by using special properties of the shape function space of Morley element and Poincaré inequality (we refer to [12, 32]), we derive an explicit bound of its interpolation error for arbitrary triangular meshes. As usual, the consistency error for plate bending problems involves some boundary residual integrals. The standard arguments to bound these terms make use of scaling arguments and trace theorems, thus the regularity assumption on the mesh can not be avoided. Our essential idea in the estimate of the consistency error is to transform some boundary integrals to some element's ones, while some approximation properties are still retained. To this end, we firstly rearrange these nonconforming terms. Then motivated by the ideas from [2], we derive an optimal estimate of the consistency error (cf. section 3 for details) with the aid of the lowest order Raviart-Thomas interpolation operator [35]. Furthermore, the optimal convergence rate of the rotation error (discrete H^1-norm) is also obtained for convex polygonal domains. The analysis carried out in this paper is made for two dimensional Morley elements, the extension to three dimensional Morley elements from [41] is also valid following the same types of arguments.

The outline of the paper is as follows. In the next section, after introducing the nonconforming Morley element approximation of the plate bending problem, we derive the interpolation error for arbitrary triangular meshes. In section 3, we mainly discuss the moment error and angular error of Morley element on meshes satisfying the maximal angle condition. In order to verify the validity of our theoretical analysis, some numerical experiments are carried out in section 4.

2. Discretization of the model problem and the interpolation error

We consider the plate bending problem:

$$\begin{cases} \triangle^2 u = f, & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω denotes a plane polygonal domain, $f \in L^2(\Omega)$ is the applied force, $n = (n_x, n_y)$ is the unit outward normal vector along the boundary $\partial\Omega$. The related variational form is :

$$\begin{cases} \text{Find } \ u \in H_0^2(\Omega), \ \text{ such that} \\ a(u,v) = (f,v), \qquad \forall v \in H_0^2(\Omega), \end{cases} \tag{2.2}$$